

THE LATTER RESEARCH ON EULER-MASCHERONI CONSTANT. A SURVEY

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Abstract. *In this survey, we aim to present historical turning points and recent insightful results on the analysis of the Euler-Mascheroni constant.*

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1. INTRODUCTION

The famous Euler-Mascheroni constant, denoted as γ , occupies a unique position in the taxonomy of fundamental mathematical constants.

$$\gamma = \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \frac{1}{k} - \ln n \right) \approx 0.5772156649$$

While π and e have long been rigorously characterized (their irrationality and transcendence were established in the 18th and 19th centuries, respectively) γ remains unknown. We currently lack a formal proof to determine if γ is a simple rational fraction or an infinite, non-repeating sequence of digits. Despite this foundational mystery, the constant appears with regularity across the mathematical landscape, from the distribution of prime numbers to the vibrations of quantum membranes.

2. CHRONOLOGY OF ANALYTIC DEVELOPMENTS

The provenance of the constant lies in the work of Leonhard Euler, who first identified it as a byproduct of his investigations into the functional equations of the Gamma function and the harmonic series [1]. In his 1734 treatise, “*De progressionibus harmonicis observationes*”, Euler provided the first decimal approximation [2]

$$C \approx 0.577218$$

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deriving it from the expansion of the digamma function $\psi(x)$.

His work established γ not as an isolated numerical curiosity, but as a critical parameter in the analytic extension of Dirichlet series.

The Mascheroni Refinement and Formalization

In 1790, Lorenzo Mascheroni published "Adnotationes ad calculum integralem Euleri", in which he extended the known precision to 32 decimal places [3].

$$C \approx 0.57721566490153286060651209008240.$$

Beyond numerical verification, Mascheroni's primary contribution was the formalization of the constant as a primary object of study in integral calculus:

- The expansion of the exponential integral, $Ei(x)$ defined for $x > 0$:

$$Ei(x) = \int_{-\infty}^x \frac{e^t}{t} dt = \gamma + \ln x + \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!}$$

- The logarithmic integral identity, $li(x)$ as $x \rightarrow 1^+$:

$$li(x) = \int_0^x \frac{dt}{\ln t} = \gamma + \ln(\ln x) + \sum_{n=1}^{\infty} \frac{(\ln x)^n}{n \cdot n!}$$

The subsequent adoption of the symbol γ served to standardize its role within the emerging field of complex analysis [4].

The Analytic Expansion of the 19th Century

During the 19th century, the work of Gauss, Dirichlet, and Riemann integrated γ into the core of analytic number theory. It became obvious that γ was intrinsically linked to the behavior of the Riemann zeta function

$$\zeta(s) = \frac{1}{s-1} + \gamma + \mathcal{O}(s-1)$$

near the pole $s = 1$. This period marked the transition of γ from a limit of a series to a fundamental constant governing the error terms in the Prime Number Theorem and the properties of L – functions [2].

The Computational Turn and Algorithmic Progress

The late 20th and early 21st centuries have been characterized by an exponential increase in computational precision. Beginning with Sweeney's 1963 electronic calculation, the pursuit of γ 's digits have necessitated the development of sophisticated algorithms, including binary splitting and high-order iterations of the Brent-McMillan method [5], which relies on the Bessel functions $I_0(x)$ and $K_0(x)$ to provide an exponentially convergent series for γ .

$$\gamma = \frac{S_0(2x) - K_0(2x)}{I_0(2x)} - \ln x$$

These computational milestones provide essential empirical data for evaluating the normality of the distribution of its digits and testing conjectures about its continued fraction expansion.

3. RECENT DEVELOPMENTS ON THE EULER-MASCHERONI CONSTANT

The primary obstacle when questioning γ 's irrationality appears to be the existence of highly complex and rapidly converging sequences that define its structure.

In recent years, these sophisticated series and sequences have become the focal point of intense study as mathematicians attempt to determine the constant's true arithmetic nature.

Considering the sequence

$$\gamma_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n$$

with γ as its limit, the search for the Euler-Mascheroni constant has moved beyond simple limits to the optimization of fast-converging series. The central challenge is that γ converges too slowly in its standard harmonic form, which leads scientists to numerous approximations for

$$\gamma_n - \gamma [6,7].$$

Starting with Tims and Tyrrell's (1971) [8] approach

$$\frac{1}{2(n+1)} < \gamma_n - \gamma < \frac{1}{2(n-1)},$$

Moving to Young (1991) [9]

$$\frac{1}{2(n+1)} < \gamma_n - \gamma < \frac{1}{2n}$$

and Tóth (1991) [10]

$$\frac{1}{2n + \frac{2}{5}} < \gamma_n - \gamma < \frac{1}{2n + \frac{1}{3}},$$

Followed by Anderson (1995) [11]

$$\frac{1-\gamma}{n} < \gamma_n - \gamma < \frac{1}{2n}, n \geq 1.$$

Then Negoï (1997) [12] introduced the sequence

$$T_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log\left(n + \frac{1}{2} + \frac{1}{24n}\right)$$

which is convergent to γ at the rate n^{-3} .

Negoi [12] then proved

$$\frac{1}{48(n+1)^3} < \gamma - T_n < \frac{1}{48n^3}$$

Following, Chen and Mortici [13] found the best possible constants

$$a \approx 0.27380525.. \text{ and } b \approx 0.2305555..$$

for the following form

$$\frac{1}{48(n+a)^3} < \gamma - T_n < \frac{1}{48(n+b)^3}.$$

Moving to the Vernescu (1999) [14] sequence

$$V_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} + \frac{1}{2n} - \log n,$$

which offered a stable n^{-2} convergence rate.

DeTemple [15,16] also introduced a sequence with a convergence rate of n^{-2}

$$R_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log\left(n + \frac{1}{2}\right)$$

as

$$\frac{1}{24(n+1)^2} < R_n - \gamma < \frac{1}{24n^2}.$$

Building on that, Chen found the best possible constants [17]

$$a = \frac{1}{\sqrt{24(-\gamma+1-\ln \frac{3}{2})}} \text{ and } b = \frac{1}{2} \text{ so that } \frac{1}{24(n+a)^2} < R_n - \gamma < \frac{1}{24(n+b)^2}.$$

Later, Chen and Mortici refined those bounds [13] to

$$\begin{aligned} \frac{1}{24\left(n + \frac{1}{2}\right)^2} - \frac{7}{960\left(n + \frac{1}{2}\right)^4} + \frac{31}{8064\left(n + \frac{1}{2}\right)^6} - \frac{127}{30720\left(n + \frac{1}{2}\right)^8} &< R_n - \gamma \\ &< \frac{1}{24\left(n + \frac{1}{2}\right)^2} - \frac{7}{960\left(n + \frac{1}{2}\right)^4} + \frac{31}{8064\left(n + \frac{1}{2}\right)^6}. \end{aligned}$$

Later, Mortici introduced a sequence [18] with a speed of rate of n^{-3}

$$\mu_n(a, b) = 1 + \frac{1}{2} + \dots + \frac{1}{n-1} + \frac{1}{an} - \ln(n+b)$$

which works for

$$a = 6 - 2\sqrt{6} \text{ and } b = \frac{1}{\sqrt{6}} \text{ or } a = 6 + 2\sqrt{6} \text{ and } b = -\frac{1}{\sqrt{6}}$$

Recently, Cringanu [7] constructed the sequence

$$L_n = 1 + \frac{1}{2} + \dots + \frac{1}{n} - \log\left(n + \frac{1}{2} + \frac{1}{24n} - \frac{1}{48n^2} + \frac{23}{5760n^3}\right), n \geq 1$$

that converges to the Euler-Mascheroni constant even faster, at a rate of n^{-5} .

A major milestone in the past year is the work of Mortici [18], who developed new sequences that converge to γ with significantly higher orders of speed.

Most recently, Mortici and Drinceanu [19] rewrote γ on the following form

$$\gamma = \sum_{n=1}^{\infty} \left(\frac{1}{n} + \ln \frac{n}{n+1} \right)$$

which is of order n^{-2} since:

$$\lim_{n \rightarrow \infty} \left(\frac{1}{n} + \ln \frac{n}{n+1} \right) n^2 = \frac{1}{2} \neq 0.$$

Then they proved by using partial sums that the following series

$$\sum_{n=1}^{\infty} \left(\frac{1}{n + \frac{1}{2}} + \ln \frac{n}{n+1} \right)$$

converges faster than the γ 's classical form. Its order is n^{-3} since:

$$\lim_{n \rightarrow \infty} \left(\frac{1}{n + \frac{1}{2}} + \ln \frac{n}{n+1} \right) n^3 = -\frac{1}{12} \neq 0.$$

Another series that they proposed, based on the initial one, where n has been replaced by $n-1$ leading to a simpler form

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} + \ln \frac{n - \frac{1}{2}}{n + \frac{1}{2}} \right)$$

which, by using the family of sequences

$$q_n(a) = \frac{1}{n + \frac{a}{n}} + \ln \frac{n - \frac{1}{2}}{n + \frac{1}{2}}, n \geq 1$$

leads to an even better convergence than the previous one for $a = -\frac{1}{12}$. In this case $(q_n(-1/12))_{n \geq 1}$ converges to zero as n^{-5} .

4. CONCLUSION

The focus on optimizing the convergence of γ -related sequences simultaneously sharpen our theoretical tools for exploring the fundamental nature of real numbers while providing the high-speed computational accuracy required in many fields of research. The ongoing evolution of these sequences ensures that γ remains a dynamic, rather than static, element of mathematical progress.

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