

ON STATISTICALLY CONVERGENCE IN REAL SERIES

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Abstract. *In this study, the concept of statistical convergence in real number series is examined. Definitions, theorems, and specific examples related to statistical convergence of series are given. The statistical convergence of the real number series, which has been studied very little, is introduced by using the natural density function. Taking advantage of this relationship, new features are examined. Theorems written on this subject are included. The grouping and the rearrangement of the terms of statistical convergent series are examined. Concepts known for Cauchy convergence, such as the Cauchy criterion and the general term test, are considered for statistical convergence. Original theorems are written on this subject, and new examples are found. Furthermore, the statistical convergence of the series is examined using tests such as the Abel and Dirichlet tests. These tests are developed, and some examples are included. Lastly, the Leibniz test and the Mertens test used in the Cauchy product series were developed for statistical convergence.*

Keywords: *Asymptotic density; statistical convergence; statistical convergent series; convergence tests.*

Mathematics Subject Classification: *40A05; 40A35.*

1. INTRODUCTION

Statistical convergence is an important concept used in summability theory, functional analysis and statistics. The concept of statistical convergence, which was first described by H. Fast [1], generalized the known Cauchy convergence concept of the series. The relationship between this concept and the summability theory was established by Schoenberg [2]. Important features of the concept of statistical convergence in the series were examined by J. Fridy [3]. Salat [4] developed this concept by addressing the topological features of statistical convergence. Furthermore, Connor [5] revealed the importance of statistical convergence for functional analysis. Freedman and Sember [6] showed the relationship between statistics and this concept by using measure theory and number theory. Tripathy was the first to introduce the concept of statistical convergence for series [7-8]. Also, Dindos, Salat, and Toma [3] examined the topological properties of statistical convergence in the series.

Let's now recall some of the basic theorems we use. Franz Carl Joseph Mertens (1840-1927) observed that

$$\left(\sum_{n=1}^{\infty} a_n\right)\left(\sum_{n=1}^{\infty} b_n\right) = \left(\sum_{n=1}^{\infty} c_n\right), \text{ where } c_n = \sum_{k=1}^n a_{n-k+1} b_k$$

still holds when only the first two series, say $\sum_{n=1}^{\infty} a_n$, converges absolutely, as long as the second $\sum_{n=1}^{\infty} b_n$ converges conditionally.

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Theorem 1.1. (Mertens). If(i) the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent and $\sum_{n=1}^{\infty} a_n = s$,(ii) the series $\sum_{n=1}^{\infty} b_n$ is convergent and $\sum_{n=1}^{\infty} b_n = t$, and(iii) $c_n = \sum_{k=1}^n a_{n-k+1} b_k$, $k \in \mathbb{N}$;then the Cauchy product series $\sum_{n=1}^{\infty} c_n$ is convergent and $\sum_{n=1}^{\infty} c_n = st$.

Abel's partial summation formula, which has been expressed below, is the discrete analogue of integration by parts.

Theorem 1.2. (Abel's partial summation formula). Let (a_n) and (b_n) be real(or complex) sequences and define the partial sums $r_p = \sum_{k=1}^p b_k$ with $p \in \mathbb{N}$, the set of all natural numbers. Then, for $p \in \mathbb{N}$

$$\sum_{k=1}^p a_k b_k = r_p a_p - \sum_{k=1}^{p-1} r_k (a_{k+1} - a_k) = r_p a_p + \sum_{k=1}^{p-1} r_k \Delta a_k,$$

where $\Delta a_k = a_k - a_{k+1}$.

The statistical versions of these two theorems are within the scope of this study.

2. MAIN RESULTS

2.1. FUNDAMENTAL DEFINITIONS AND PROPERTIES

The fact that a series $\sum_{k=1}^{\infty} a_k$ is statistically convergent means that its partial sums sequence by the terms $s_n = \sum_{k=1}^n a_k$ is statistically convergent. The idea is based on the asymptotic density of a subset A of the natural numbers set. This density is denoted by $\delta(A)$ and is defined by $\delta(A) = \lim_{n \rightarrow \infty} \frac{|A(n)|}{n}$ with $A(n) = \{k \leq n: k \in A\}$ for each natural number n , where $|\cdot|$ denotes the size(cardinality) of a set.

Definition 2.1.1. Let $\sum_{k=1}^{\infty} a_k$ be a real number series and (s_n) be its partial sums sequence. If, for all $\varepsilon > 0$, $\lim_{n \rightarrow \infty} \frac{|\{k \leq n: |s_k - L| \geq \varepsilon\}|}{n} = 0$, then it is said that the partial sums sequence (s_n) is statistically convergent to L , and is denoted by $s_n \rightarrow L$ (*stat*) or $st - \lim_n s_n = L$.

Definition 2.1.2. Let $\delta(K) = 0$ with $K \subseteq \mathbb{N}$. Given any $\varepsilon > 0$, if there is a natural number n_0 such that $|a_n - L| < \varepsilon$ for $n \geq n_0$ and $n \notin K$, then the sequence $a = (a_n)$ is called convergent to L for almost every n .

This definition, given by Buck [10], is equivalent to the definition of statistical convergence and can be used to establish statistical convergence under favourable conditions.

Definition 2.1.3. If there is at least one $A > 0$ such that $\delta[\{k \in \mathbb{N}: |s_k| > A\}] = 0$, then it is said that the sequence (s_k) is statistically bounded.

The following theorem was given by Tripaty [8].

Theorem 2.1.1. If the series $\sum_{k=1}^{\infty} a_k$ is statistical convergent, then there exists a subset $K = \{k_1 < k_2 < k_3 < \dots\}$ of natural numbers such that $\delta(K) = 1$ and the series $\sum_{i=1}^{\infty} a_{k_i}$ is convergent.

The opposite of the above theorem is not always true, for example, if $a_n = \begin{cases} 1, & n = m^2 \\ \frac{1}{n^2}, & n \neq m^2 \end{cases}$ with $m \in \mathbb{N}$, then $\delta(K) = 1$ with $K = \{n : n \neq m^2, m \in \mathbb{N}\}$. Again, although the series $\sum_{n \in K} a_n$ is convergent, $\sum_{n=1}^{\infty} a_n$ is not statistically convergent.

The following two propositions were given by Tripaty [8].

Proposition 2.1.1. The necessary and sufficient condition for a real number series with nonnegative terms to be statistically convergent is that this series is convergent.

Proposition 2.1.2 If the series $\sum_{n=1}^{\infty} a_n$ is statistically convergent, then $a_n \rightarrow 0$ (stat).

The opposite of the last proposition is not always true.

Example 2.1.1. For all $n \in \mathbb{N}$, let

$$a_n = \begin{cases} e^n, & n = m^2 \\ \frac{1}{n^2}, & n \neq m^2, m \in \mathbb{N}. \end{cases}$$

Then, although $a_n \rightarrow 0$ (stat), the series $\sum_{n=1}^{\infty} a_n$ is not statistically convergent. Because the series $\sum_{n=m^2} a_n$ with $m \in \mathbb{N}$ is divergent, and this makes the series statistically divergent. $\sum_{n=1}^{\infty} a_n$.

Remark 2.1.1. While $\sum_{n=1}^{\infty} a_n$ is a statistical convergent series, even if $a_n \rightarrow 0$, this series may not be Cauchy convergent. For example, this claim holds for the series $\sum_{n=1}^{\infty} a_n$ with

$$a_n = \begin{cases} \frac{1}{\sqrt{n}}, & n = m^2 \\ -1, & n = m^2 + 1, \\ \frac{1}{n\sqrt{n}}, & otherwise \end{cases} \quad m \in \mathbb{N}$$

for all $n \in \mathbb{N}$. Because it is obviously $a_n \rightarrow 0$, and if the series $\sum_{n=1}^{\infty} a_n$ is considered, then

$$\sum_{m \in \mathbb{N}, n=m^2} \frac{1}{\sqrt{n}} = \sum_{m \in \mathbb{N}} \frac{1}{m}$$

is divergent as a harmonic series and $a_{m^2} + a_{m^2+1} = 0$ for all $m \in \mathbb{N}$, hence, although the partial sums sequence (s_n) is statistically convergent but not Cauchy convergent.

The following proposition was given by Tripaty [8].

Proposition 2.1.3. The series $\sum_{n=1}^{\infty} a_n$ is statistically convergent if and only if, for any given $\varepsilon > 0$, there is at least one $m_0 \in \mathbb{N}$ such that

$$\lim_{n \rightarrow \infty} \frac{|\{n \in \mathbb{N} : |\sum_{k=m}^n a_k| \geq \varepsilon\}|}{n} = 0,$$

where $n \geq m \geq m_0$.

Proof: If the series $\sum_{n=1}^{\infty} a_n$ is statistically convergent, then its partial sums sequence (s_n) is a statistical Cauchy sequence. In other words, there are a number $N \in \mathbb{N}$ and a subset $K \subset \mathbb{N}$ such that $\delta(K) = 1$ and $|s_n - s_N| < \varepsilon$ for all $n \in K$. If $N = m - 1$ is selected, then

$$|s_n - s_{m-1}| = \left| \sum_{k=1}^n a_k - \sum_{k=1}^{m-1} a_k \right| = \left| \sum_{k=m}^n a_k \right|.$$

Since $\delta(K) = 1$ and $\delta(\mathbb{N} - K) = 0$, it is obtained $\delta(\{n \in \mathbb{N}: |\sum_{k=m}^n a_k| \geq \varepsilon\}) = 0$. Conversely, for given any $\varepsilon > 0$, we accept that there is at least one $m_0 \in \mathbb{N}$ such that

$$\lim_{n \rightarrow \infty} \frac{|\{n \in \mathbb{N}: |\sum_{k=m}^n a_k| \geq \varepsilon\}|}{n} = 0,$$

where $n \geq m \geq m_0$. This shows that

$$\delta\left(\left\{n \in \mathbb{N}: \left| \sum_{k=1}^n a_k - \sum_{k=1}^{m-1} a_k \right| \geq \varepsilon \right\}\right) = \delta(\{n \in \mathbb{N}: |s_n - s_{m-1}| \geq \varepsilon\}) = 0.$$

Then it can be seen that the partial sums sequence (s_n) be a statistical Cauchy sequence, whenever $m = N + 1$ is taken. The following theorem was given by Tripaty [7].

Theorem 2.1.2. If the series $\sum_{k=1}^{\infty} a_k$ is statistically convergent and (ka_k) is a sequence in ℓ_{∞} , the space of bounded real sequences, then $\sum_{k=1}^{\infty} a_k$ is Cauchy convergent.

The opposite of this theorem is not true in general. Indeed, according to the Leibniz criterion, the series $\sum_{n=1}^{\infty} (-1)^k \frac{1}{\log k+1}$ is Cauchy convergent, since the sequence $\left(\frac{1}{\log k+1}\right)$ is monotone decreasing and $\lim_{k \rightarrow \infty} \frac{1}{\log k+1} = 0$, but the sequence (ka_k) with $a_k = \frac{1}{\log k+1}$ is not bounded.

Remark 2.1.2. If real series $\sum_{k=1}^{\infty} a_k$ is statistical convergent, then this series is statistical bounded but vice versa. For example, if

$$a_k = \begin{cases} k, & k = m^3 \\ 1 - k, & k = m^3 + 1, \\ (-1)^k, & \text{otherwise} \end{cases} \quad m \in \mathbb{N}$$

for all $k \in \mathbb{N}$, then the sequence (a_k) satisfies this assertion.

Let (s_k) be the sequence of partial sums of the series $\sum_{k=1}^{\infty} a_k$. The subsequence (s_{m^3}) of (s_k) is unbounded. Also, since $a_{m^3} + a_{m^3+1} = 0$ for all $m \in \mathbb{N}$, the subsequence (s_{m^3+1}) is bounded. In other cases, since the sum of the general term $(-1)^k$ is bounded, it is clear that there exists a $K \subseteq \mathbb{N}$ such that $\delta(K) = 1$. Thus, the series $\sum_{k=1}^{\infty} a_k$ is statistically bounded, but this series is under the control of the general term $(-1)^k$ in the set $K = \{k | k \neq m^3, k \neq m^3 + 1, m \in \mathbb{N}\}$ with natural density 1, hence the series is not statistically convergent.

Remark 2.1.3. Let P be the set of prime numbers. If real series $\sum a_k$ is statistically convergent, then the sequence (a_k) does not have to be bounded. For example, although the sequence a_k with

$$a_k = \begin{cases} \log k, & k = p \\ \log \frac{1}{k-1}, & k = p + 1, \quad p \in P, \\ \frac{1}{k^2}, & \text{otherwise} \end{cases}$$

is unbounded for all $k \in \mathbb{N}$, the series $\sum a_k$ is statistical convergent.

Indeed, since $a_p + a_{p+1} = 0$ for all $p \in P$ and the series $\sum_{k \neq p^2, k \neq p^2+1} \frac{1}{k^2}$ is convergent, the series $\sum_{k=1}^\infty a_k$ is statistically convergent, but the sequence a_k is clearly unbounded. The following theorem was presented by Tripathy [8].

Theorem 2.1.3. The necessary and sufficient condition to have a weak subseries diverging to $+\infty$ of statistical convergence series $\sum a_k$ is that this series has a weak subseries diverging to $-\infty$.

Theorem 2.1.4. Let (a_k) be a real sequence and the series $\sum_{k \in K} a_k$ be convergent and its sum L with $K \subseteq \mathbb{N}$ and $\delta(K) = 1$, and be $\sum_{k \in K} a_k = L$. If $m_i \in M, s_i \in S$ and $m_i = c + s_i$ for a fixed $c \in \mathbb{N}$ and the classified series $\sum_{i \in \mathbb{N}} (a_{m_i} + a_{s_i})$ is convergent and its sum is R , where $\mathbb{N} \setminus K = M \cup S, M \cap S = \emptyset$ with $M = \{m_1 < m_2 < \dots < m_i \dots\}$ and $S = \{s_1 < s_2 < \dots < s_i \dots\}$, then the series $\sum_{k=1}^\infty a_k$ is statistical convergent.

Proof: Let (t_p) with $t_p = \sum_{k=1}^p a_k$ be a partial sum sequence of the series $\sum_{k=1}^\infty a_k$. We decompose the partial sum t_p in the form

$$t_p = \sum_{\substack{k \leq p \\ k \in K}} a_k + \sum_{\substack{k \leq p \\ k \in M}} a_k + \sum_{\substack{k \leq p \\ k \in S}} a_k.$$

Since the series $\sum_{k \in K} a_k$ converges and $\sum_{k \in K} a_k = L$, we write

$$\lim_{p \rightarrow \infty} \sum_{\substack{k \leq p \\ k \in K}} a_k = L.$$

Moreover, since $\delta(K) = 1$, we have $\delta(\mathbb{N} \setminus K) = \delta(M \cup S) = 0$. Again, since $\sum_{i \in \mathbb{N}} (a_{m_i} + a_{s_i}) = R$, we can write

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n (a_{m_i} + a_{s_i}) = R.$$

Let us observe that

$$\sum_{\substack{k \leq p \\ k \in K}} a_k + \sum_{\substack{k \leq p \\ k \in M}} a_k = \sum_{i=1}^{\rho(p)} (a_{m_i} + a_{s_i}),$$

where $\rho(p)$ represents the number of pairs of m_i and s_i up to p . Thus, we write

$$\sum_{k=1}^p a_k = \sum_{\substack{k \leq p \\ k \in K}} a_k + \sum_{i=1}^{\rho(p)} (a_{m_i} + a_{s_i}),$$

hence $\lim_{p \rightarrow \infty} \sum_{k=1}^p a_k = L + R$. Since the density of $M \cup S$ is zero,

$$\lim_{p \rightarrow \infty} \frac{1}{p} |\{k \leq p : p \in M \cup S\}|$$

is written, hence the expression

$$|\sum_{k=1}^p a_k - (L + R)|$$

can only show significant deviations in a set with zero density. This shows that the sequence $(\sum_{k=1}^p a_k)$ is statistical convergent to the number $(L + R)$.

Example 2.1.2. The series $\sum_{n=1}^{\infty} a_n$ with

$$a_n = \begin{cases} n, & n = m^2 \\ \frac{1-(n-8)^3}{(n-8)^2}, & n = m^2 + 8, m \in \mathbb{N}, \\ \left(\frac{1}{2}\right)^n, & \text{otherwise} \end{cases}$$

for every $n \in \mathbb{N}$, satisfies the last theorem.

2.2. REARRANGEMENT OF SERIES

It is known that a rearrangement of a conditionally convergent series can be divergent; this also holds for statistical convergence.

Example 2.2.1. Let a sequence (a_k) be defined by the terms

$$a_k = \begin{cases} 2, & k = m^2 \\ -2, & k = m^2 + 1. \\ k^{-2}, & \text{others} \end{cases}$$

Since $a_{m^2} + a_{m^2+1} = 0$ for all $m \in \mathbb{N}$ and the series $\sum_{k \in K} \frac{1}{k^2}$ converges with $K = \{k | k \neq m^2, k \neq m^2 + 1, m \in \mathbb{N}\}$ and $\delta(K) = 1$, the series $\sum_{k=1}^{\infty} a_k$ is statistical convergent. Let us rearranged this series as

$$\sum_{k=1}^{\infty} a_k = 2 + \frac{1}{9} - 2 + 2 + \frac{1}{36} - 2 + 2 + \frac{1}{49} - 2 + \dots$$

When we constantly bring $\frac{1}{k^2}$ for all $k \in K$ between the terms 2 and -2 , the new series cannot be Cauchy convergent in a set with natural intensity 1, because, after a sufficiently large k value, the values of 2 and -2 will distort convergence in all three consecutive terms. The convergent series cannot be statistically convergent, since Cauchy convergence can fail.

In addition, when we consider some of the series that are not statistically convergent, these series can be ensured to be statistically convergent by a proper rearrangement of their terms.

Example 2.2.2. Let a sequence a_k be defined in the form

$$a_k = \begin{cases} (-1)^k, & k = m^2 \\ (-1)^k, & k = m^2 + 1 \\ 2^{-k}, & \text{otherwise} \end{cases}$$

for all $m \in \mathbb{N}$. Although the series $\sum_{k \in K} 2^{-k}$ converges with

$$K = \{k | k \neq m^2, k \neq m^2 + 1, m \in \mathbb{N}\}$$

and $\delta(K) = 1$, the series $\sum_{k=1}^{\infty} a_k$ cannot be statistical converge to a fixed value L in a set with a density of 1, because of the sum $a_{m^2} + a_{m^2+2} = \begin{cases} 2, & \text{if } m \text{ is even} \\ -2, & \text{if } m \text{ is odd} \end{cases}$. Let us try to rearrange the terms of the series appropriately. For every odd number m , when we change the terms a_{m^2} and $a_{(m+1)^2}$, the series in the form

$$\sum_{k=1}^n a_k = 1 + \frac{1}{4} - 1 - 1 + \frac{1}{32} + 1 + \frac{1}{128} + \frac{1}{256} + 1 + \frac{1}{1024} - 1 + \dots + \frac{1}{2^{n-4}} + \frac{1}{2^{n-3}} - 1 + \frac{1}{2^{n-1}} + 1$$

with $n = r^2 + 2$ for an odd number r reveals and since this series converges in the Cauchy sense in a density of 1 and $a_{m^2} + a_{m^2+2} = 0$ for all $m \in \mathbb{N}$, it is statistically convergent to a number L .

2.3. GROUPING IN SERIES

If a real number series is Cauchy convergent to a real number L , then the new series obtained by grouping the terms of this series is also Cauchy convergent to the same real number L . For statistically convergent series, the analysis is much more difficult because, when these series contain any signed terms, they may not be convergent in the Cauchy sense.

Theorem 2.3.1 If a real number series $\sum_{k=1}^{\infty} a_k$ is statistically convergent, then the new series formed by grouping this series is also statistically convergent to the same number.

Proof: Let the real series $\sum_{k=1}^{\infty} a_k$ be statistically convergent to L . In this case, there exists a subset $K = \{k_1 < k_2 < k_3 < \dots\} \subset \mathbb{N}$ with $\delta(K) = 1$ such that $\lim_{n \rightarrow \infty} s_{k_n} = L$ with $\sum_{k=1}^n a_k = s_n$. Any grouping of the series $\sum_{k=1}^{\infty} a_k$ is actually a subsequence of the partial sums sequence s_n and since $\lim_{n \rightarrow \infty} s_{k_n} = L$ when we accept this subsequence s_{k_n} , in fact, the series created by grouping the first series appears to be converging in the Cauchy sense to a number L in a set K with a density of 1. In this case, the sums A and \bar{A} are equal to each other, where $\sum_{k=1}^{\infty} a_k = A$ and the sum of the series formed by grouping the terms without changing the order is \bar{A} .

Example 2.3.1. Let $a_n = \begin{cases} n & , n = 2m^2 - 1 \\ 1 - n & , n = 2m^2 \\ 2^{-n} & , \text{otherwise} \end{cases}$ for all $n \in \mathbb{N}$ with $m \in \mathbb{N}$. Let us consider

the series $\sum_{n=1}^{\infty} a_n$. Since the series $\sum_{k \in K} a_k$ is convergent in the set $K = \{k \mid k \neq 2m^2, k \neq 2m^2 - 1, m \in \mathbb{N}\}$ with $\delta(K) = 1$ and $a_{2m^2} + a_{2m^2-1} = 0$, the series $\sum_{n=1}^{\infty} a_n$ is convergent in the Cauchy sense on K and thus statistically convergent.

Let us group this series two by two. In this case, if $k = 2m^2$ for some $m \in \mathbb{N}$, then a grouping in the form

$$\sum_{n=1}^k (a_{2n-1} + a_{2n}) = (1 - 1) + \left(\frac{1}{8} + \frac{1}{16}\right) + \left(\frac{1}{32} + \frac{1}{64}\right)$$

$$+(7-7)+\dots+\left(\frac{1}{2^{k-3}}+\frac{1}{2^{k-2}}\right)+(k-1+1-k)$$

occurs, and since the term 0 is consistently derived from the set with a density of 0. It does not affect its total value, and this series converges in the Cauchy sense.

If we group the terms of this series three by three, the series in the form

$$\begin{aligned} \sum_{n=1}^k (a_{3n-2} + a_{3n-1} + a_{3n}) &= \left(1 - 1 + \frac{1}{8}\right) + \left(\frac{1}{16} + \frac{1}{32} + \frac{1}{64}\right) \\ &+ \left(7 - 7 + \frac{1}{512}\right) + \dots + \left(\frac{1}{2^{k-2}} + \frac{1}{2^{k-1}} + k\right) + \dots \end{aligned}$$

occurs, where $k = 2m^2 - 1$ for a large enough natural number m . This series is divergent, but this only occurs when $k = 2m^2 - 1$ for all $m \in \mathbb{N}$ and also $\delta(A) = 0$ with $A = \{k | k = 2m^2 - 1, m \in \mathbb{N}\}$. In the set A with a density of 1, this series will converge in the Cauchy sense, and, in general, the group series of this series are statistically convergent.

2.4. SOME PROPERTIES AND TESTS FOR STATISTICAL CONVERGENCE OF ANY SIGNED REAL NUMBER SERIES

The fact that the statistical convergence of positive-term real number series coincided with Cauchy convergence made it unnecessary to develop tests for positive-term series. For this reason, this section will examine the statistical convergence of any signed real number series.

Theorem 2.4.1. Let (a_n) and (b_n) be two sequences of real numbers, and the following two conditions come true:

- (i) The sequence (a_n) is bounded and statistically convergent to zero.
- (ii) The sequence $\sum_{i=1}^n |b_i|$ is bounded.

Then, the sequence $\frac{1}{n} \sum_{i=1}^n a_i b_i$, Cesaro means of the sequence $a_n b_n$, converges to zero.

Proof: If the sequences (a_n) and $(\sum_{i=1}^n |b_n|)$ are bounded, then there is a number $K > 0$ such that $|a_n| < K$ and $\sum_{i=1}^n |b_i| < K$ for every $n \in \mathbb{N}$. If it is used, the sequence (a_n) is statistical convergence to zero, then, given any $\varepsilon > 0$, there exists at least one $n_\varepsilon \in \mathbb{N}$ such that

$$\frac{1}{n} \left| \left\{ k \leq n : |a_k| \geq \sqrt{\frac{\varepsilon}{2}} \right\} \right| < \frac{\varepsilon}{2K^2}$$

for every $n \geq n_\varepsilon$. We can write $\frac{A(n)}{n} < \frac{\varepsilon}{2K^2}$ for every $n \geq n_\varepsilon$ with $A(n) = \left| \left\{ k \leq n : |a_k| \geq \sqrt{\frac{\varepsilon}{2}} \right\} \right|$. Hence, we have

$$\begin{aligned} \frac{1}{n} \sum_{k=1}^n |a_k|^2 &= \frac{1}{n} \left(\sum_{1 \leq k \leq n, |a_k| \geq \sqrt{\frac{\varepsilon}{2}}} |a_k|^2 + \sum_{1 \leq k \leq n, |a_k| < \sqrt{\frac{\varepsilon}{2}}} |a_k|^2 \right) \\ &\leq \frac{1}{n} \left(A(n)K^2 + n \frac{\varepsilon}{2} \right) < \frac{A(n)}{n} K^2 + \frac{\varepsilon}{2} < \frac{\varepsilon}{2K^2} K^2 + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

for every $n \geq n_\varepsilon$. Therefore, the sequence $\left(\frac{1}{n} \sum_{k=1}^n |a_k|^2\right)$ is ordinary convergent to zero. Since the sequence $\left(\sum_{k=1}^n |b_k|\right)$ is bounded, the sequence (b_k) is bounded and thus the sequence $\left(\sum_{k=1}^n b_k^2\right)$ is bounded. Thus, there is one $M > 0$ such that $\sum_{k=1}^n (b_k)^2 < M$ for every $n \in \mathbb{N}$. Since the sequence $\left(\frac{1}{n} \sum_{k=1}^n |a_k|^2\right)$ converges to zero, given $\varepsilon > 0$, there is a natural number n_ε such that $\frac{1}{n} \sum_{k=1}^n |a_k|^2 < \frac{\varepsilon^2}{M}$ for every $n \geq n_\varepsilon$. Also, by Hölder's inequality

$$\frac{1}{n} \left| \sum_{k=1}^n a_k b_k \right| \leq \frac{1}{n} \sum_{k=1}^n |a_k b_k| \leq \frac{1}{n} \sqrt{\sum_{k=1}^n |a_k|^2} \sqrt{\sum_{k=1}^n |b_k|^2} < \frac{\varepsilon}{\sqrt{M}} \sqrt{M} = \varepsilon$$

and hence the sequence $\left(\frac{1}{n} \sum_{k=1}^n a_k b_k\right)$ is convergent to zero.

At this stage, we will present a version of Mertens' theorem that uses statistical convergence.

Theorem 2.4.2 Let (a_n) and (b_n) be two sequences of real numbers and the following two conditions come true:

- (i) $s_n \rightarrow s$ and the series $\sum_{n=1}^\infty (s_n - s)$ is absolutely convergent, where $s_n = \sum_{k=1}^n a_k$.
- (ii) The series $\sum_{n=1}^\infty b_n$ is statistically convergent, $b_n \rightarrow 0$ and $\sum_{k=1}^n b_k \rightarrow t$ (stat), as $n \rightarrow \infty$.

Then, $u_n \rightarrow st$ (stat) with $c_n = \sum_{k=1}^n a_{n-k+1} b_k$ and $u_n = c_1 + c_2 + \dots + c_n$.

Proof: Let $s_n = \sum_{k=1}^n a_k$, $p_n = \sum_{k=1}^n b_k$ and $u_n = \sum_{k=1}^n c_k$ for all natural numbers n . Assume $r_n = s_n - s$ for all n . Since the series $\sum_{n=1}^\infty r_n$ is absolutely convergent, there is a number $M > 0$ such that $\sum_{k=1}^n |r_k| \leq M$ for every n . We obtain

$$\begin{aligned} u_n &= a_1 b_1 + (a_2 b_1 + a_1 b_2) + \dots + (a_n b_1 + a_{n-1} b_2 + \dots + a_1 b_n) \\ &= b_1 (a_1 + \dots + a_n) + b_2 (a_1 + \dots + a_{n-1}) + \dots + b_n a_1 \\ &= b_1 s_n + b_2 s_{n-1} + \dots + b_n s_1 \\ &= b_1 (r_n + s) + b_2 (r_{n-1} + s) + \dots + b_n (r_1 + s) \\ &= s (b_1 + b_2 + \dots + b_n) + (b_1 r_n + b_2 r_{n-1} + \dots + b_n r_1) \end{aligned}$$

for every $n \in \mathbb{N}$. According to this, we can write $u_n = s.p_n + \beta_n$ with $\beta_n = b_1 r_n + b_2 r_{n-1} + \dots + b_n r_1$ for every $n \in \mathbb{N}$. Since $b_n \rightarrow 0$, given any $\varepsilon > 0$, there is a natural number n_ε such that $|b_n| < \frac{\varepsilon}{2M}$ for every $n \geq n_\varepsilon$, and a number $R > 0$ such that $|b_n| \leq R$ for every n . In this case, we obtain

$$|\beta_n| \leq |b_1 r_n + b_2 r_{n-1} + \dots + b_{n_\varepsilon} r_{n-n_\varepsilon+1}| + |b_{n_\varepsilon+1} r_{n-n_\varepsilon} + \dots + b_n r_1|$$

$$\leq R(|r_n| + |r_{n-1}| + \dots + |r_{n-n_\varepsilon+1}|) + \frac{\varepsilon}{2M}(|r_{n-n_\varepsilon}| + \dots + |r_1|).$$

Since the series $\sum_{n=1}^{\infty} r_n$ is absolutely convergent, using Cauchy criterion for the convergent series, we have there exists a natural number N such that $|r_n| + |r_{n-1}| + \dots + |r_{n-n_\varepsilon+1}| < \frac{\varepsilon}{2R}$ with $n \geq N > n_\varepsilon$ and $n - n_\varepsilon > N$. Therefore

$$|\beta_n| \leq R \frac{\varepsilon}{2R} + \frac{\varepsilon}{2M} M = \varepsilon,$$

and this shows that $\beta_n \rightarrow 0$. Since $u_n = s.p_n + \beta_n$ and $p_n \rightarrow t$ (stat) we reach the result $u_n \rightarrow st$ (stat).

Example 2.4.1. Let the sequences (a_n) and (b_n) be given as follows:

$$a_n = (-1)^n \left(\frac{1}{(n+1)^2} + \frac{1}{n^2} \right) \text{ and } b_n = \begin{cases} \frac{1}{\sqrt{n}}, & n = m^2 \\ -\frac{1}{\sqrt{n-1}}, & n = m^2 + 1, \\ \frac{1}{n\sqrt{n}}, & \text{otherwise} \end{cases} \quad m \in \mathbb{N}.$$

Then, the series $\sum_{n=1}^{\infty} a_n$ is convergent due to Leibniz's criterion. Assuming $s_n = \sum_{k=1}^n a_k$ for all $n \in \mathbb{N}$, we have $s_n = 1 + (-1)^{n+1} \frac{1}{(n+1)^2}$ for all $n \in \mathbb{N}$ and hence $\lim_{n \rightarrow \infty} s_n = s = 1$. Clearly $s_n - s = (-1)^{n+1} \frac{1}{(n+1)^2}$ and the series $\sum_{n=1}^{\infty} (s_n - s)$ is absolutely convergent. Moreover, it is easily seen that the sequence $(\sum_{k=1}^n b_k)$ is statistically convergent and $b_n \rightarrow 0$. Thus, the hypotheses of the last theorem are realized.

Theorem 2.4.3. Let (a_n) and (b_n) be two sequences of real numbers, and the following two conditions come true:

- (i) The sequence (a_n) is monotone and the sequence (c_n) is monotone and bounded with $a_n = \frac{c_n}{n}$ for every $n \in \mathbb{N}$.
- (ii) The sequence $(\sum_{k=1}^n |a_k|)$ is bounded.
- (iii) The sequence (b_n) is bounded and $b_n \rightarrow 0$ (stat).

Then, the series $\sum_{n=1}^{\infty} a_n b_n$ is convergent.

Proof: Since the sequences $(\sum_{k=1}^n |a_k|)$, (b_n) and (c_n) are bounded, there are positive numbers L , K and M with $\sum_{k=1}^n |a_k| < L$, $|b_n| < K$ and $|c_n| < M$ for every $n \in \mathbb{N}$. Also, since $b_n \rightarrow 0$ (stat),

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n: |b_k| \geq \varepsilon\}| = 0.$$

Now, let $A(n) = |\{k \leq n: |b_k| \geq \varepsilon\}|$. We have

$$\frac{1}{n} \left| \sum_{k=1}^n b_k \right| \leq \frac{1}{n} \sum_{k=1}^n |b_k| \leq \frac{1}{n} \left(\sum_{1 \leq k \leq n, |b_k| < \varepsilon} |b_k| + \sum_{1 \leq k \leq n, |b_k| \geq \varepsilon} |b_k| \right)$$

$$\begin{aligned} &\leq \frac{1}{n} \left(\sum_{1 \leq k \leq n, |b_k| < \varepsilon} \varepsilon + \sum_{1 \leq k \leq n, |b_k| \geq \varepsilon} K \right) \\ &< \frac{1}{n} (n\varepsilon + A(n)K) = \varepsilon + \frac{A(n)}{n}K \end{aligned}$$

and hence $\lim_{n \rightarrow \infty} \frac{s_n}{n} = 0$ with $s_n = \sum_{k=1}^n b_k$ for every $n \in \mathbb{N}$. For this reason, given any $\varepsilon > 0$, there is a number $n_\varepsilon \in \mathbb{N}$ such that $\left| \frac{sp}{p} \right| < \frac{\varepsilon}{6M+2L}$ for every $p \geq n_\varepsilon$. Therefore, $\left| \frac{sp}{p} - \frac{s_{n_\varepsilon}}{p} \right| = \left| \frac{r_p}{p} \right| < \frac{\varepsilon}{3M+L}$ for every $p \geq n_\varepsilon$ with $r_p = b_{n_\varepsilon+1} + \dots + b_p$. If we apply Abel's partial sum formula(Theorem 1.1 on page 1), then we have

$$\begin{aligned} \left| \sum_{k=n_\varepsilon+1}^p a_k b_k \right| &= \left| a_p r_p + \sum_{k=n_\varepsilon+1}^{n-1} (a_k - a_{k+1}) r_k \right| \leq |a_p| |r_p| + \sum_{k=n_\varepsilon+1}^{n-1} |a_k - a_{k+1}| |r_k| \\ &\leq \left| \frac{c_p}{p} \right| |r_p| + \sum_{k=n_\varepsilon+1}^{n-1} \left| \frac{c_k}{k} - \frac{c_{k+1}}{k+1} \right| |r_k| = |c_p| \left| \frac{r_p}{p} \right| + \sum_{k=n_\varepsilon+1}^{n-1} \left| \frac{c_k}{k} - \frac{c_{k+1}}{k+1} \right| |r_k| \\ &\leq M \frac{\varepsilon}{3M+L} + \sum_{k=n_\varepsilon+1}^{n-1} \frac{1}{k+1} \left| \frac{k+1}{k} c_k - c_{k+1} \right| |r_k| \\ &\leq \frac{\varepsilon}{3M+L} \left(M + \sum_{k=n_\varepsilon+1}^{n-1} \left| \frac{c_k}{k} + c_k - c_{k+1} \right| \right) \\ &\leq \frac{\varepsilon}{3M+L} \left(M + \sum_{k=n_\varepsilon+1}^{n-1} \left(\left| \frac{c_k}{k} \right| + |c_k - c_{k+1}| \right) \right) \\ &\leq \frac{\varepsilon}{3M+L} \left(M + \sum_{k=n_\varepsilon+1}^{n-1} \left| \frac{c_k}{k} \right| + |c_{n_\varepsilon+1} - c_n| \right) \\ &\leq \frac{\varepsilon}{3M+L} (3M+L) = \varepsilon. \end{aligned}$$

This completes the proof.

Theorem 2.4.4. Let (a_n) and (b_n) be two sequences of real numbers, and the following three conditions are satisfied:

- (i) The sequence (a_n) is statistical bounded.
- (ii) The sequence (Δa_n) is bounded.
- (iii) The series $\sum_{k=1}^\infty s_k$ is absolutely convergent with $s_k = \sum_{i=1}^k b_i$.

Then, the series $\sum_{n=1}^\infty a_n b_n$ is statistically convergent.

Statistically the sequence (Δa_n) is bounded, then there is a number $K > 0$ such that $|\Delta a_n| < K$ for every $n \in \mathbb{N}$. Also, since the sequence (a_n) is statistical bounded, there are a subset $P \subset \mathbb{N}$ with $\delta(P) = 1$ and a number $A > 0$ such that $|a_p| < A$ for every $p \in P$. In addition to this, since the series $\sum_{k=1}^\infty s_k$ is absolutely convergent with $s_k = \sum_{i=1}^k a_i$, $\lim_{k \rightarrow \infty} s_k = 0$ and, given $\varepsilon > 0$, there exists a natural number n_1 such that $\sum_{k=n_1+1}^p |s_k| < \frac{\varepsilon}{2K}$ for every $p \geq n_1$,

and since $\lim_{k \rightarrow \infty} s_k = 0$, there exists a natural number n_2 such that $|s_k| < \frac{\varepsilon}{2A}$ for every $k \geq n_2$. Let $N = \max\{n_1, n_2\}$. Taken $r_p = b_{N+1} + \dots + b_p = \sum_{k=N+1}^p b_k$, if Abel's partial sum formula (Theorem 1.2 on page 2) is used, it is obtained

$$\begin{aligned} \left| \sum_{k=N+1}^p a_k b_k \right| &= \left| a_p r_p + \sum_{k=N+1}^{p-1} (\Delta a_k) r_k \right| \leq |a_p| |r_p| + \sum_{k=N+1}^{p-1} |\Delta a_k| |r_k| \\ &\leq A \frac{\varepsilon}{2A} + K \sum_{k=N+1}^{p-1} |r_k| < \frac{\varepsilon}{2} + K \frac{\varepsilon}{2K} = \varepsilon, \end{aligned}$$

where $p \in P$. This completes the proof.

3. CONCLUSIONS

In this study, the sequence of partial sums of a real number series was used to examine statistical convergence. By considering the general term of the series, various cases of the series' sum were observed. These observations revealed that determining the sequence of partial sums of series with a piecewise general term requires special methods. Unlike sequences, many properties were examined, and new theorems were discovered. The relationship between the statistical convergence of real-valued series and the density function was investigated. Cauchy product series were considered, and Abel, Dirichlet, Leibniz, and Mertens tests were developed for statistical convergence.

The natural (asymptotic) density function was used to examine the statistical convergence of real number series. These investigations can be performed using a different density function μ , yielding new properties.

Furthermore, the concept of statistical convergence for function series, which is more difficult to study, can be examined by developing the ideas presented in this thesis. Here, determining the density function is also crucial. Different properties and situations can be revealed with different density functions.

This study offers a different perspective on real problems in statistics, one that goes beyond the Cauchy convergence model.

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