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UNIT NEW TWO-PARAMETER ARADHANA DISTRIBUTION: THEORY, ESTIMATION AND APPLICATIONS

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Abstract. *This study introduces a new bounded distribution defined on the (0,1) interval. Various statistical properties of the proposed distribution, including moments, coefficients of skewness and kurtosis, order statistics, and stochastic ordering, are investigated. The two unknown parameters of the distribution are estimated using twelve different estimation methods, including maximum likelihood, Cramér-von Mises, Anderson-Darling, least squares, weighted least squares, maximum product of spacings, minimum spacing absolute distance, minimum spacing absolute log distance, Anderson-Darling left tail, Anderson-Darling left tail second-order, Anderson-Darling right tail and Kolmogorov method. The performance of these estimators is evaluated using a Monte Carlo simulation with 10,000 replications across various scenarios. Furthermore, to demonstrate the superiority of the proposed distribution over well-known competing bounded distributions such as the beta and Kumaraswamy distributions, it is analyzed and assessed on two real datasets.*

Keywords: *Unit distribution; maximum likelihood estimation; Monte Carlo simulation; least squares method.*

Mathematics Subject Classification: *62F10; 60E05; 62E15; 65C05; 65C10.*

1. INTRODUCTION

The advancement of data science and artificial intelligence has increased data diversity, creating a growing demand for new statistical models. Among the most fundamental types of models are discrete, continuous and bounded distributions. Although bounded distributions are constrained within a specific range, they have gained significant popularity and become essential in various disciplines, including physics, chemistry, biology, health sciences, economics, finance, and engineering.

The bounded distributions are developed over time to address various needs. The most frequently studied type of bounded distribution is the unit distribution, which is defined on the (0,1) interval. Unit distributions are typically used to model data involving proportions and percentages. Some of the most well-known unit distributions in the literature include beta (B), Topp-Leone [1], Kumaraswamy (KW) [2], unit-Gamma [3], unit-Birnbaum-Saunders [4], unit-logistic [5], unit-inverse Gaussian [6] and unit-Weibull (UW) [7] distributions.

In recent decades, unit distributions have gained popularity, often obtained by applying transformations to continuous random variables or by integrating specific families of distributions into unit distributions. A list of recently proposed unit distributions obtained through these methods is provided in Table 1.

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Table 1. Recently obtained unit distributions.

Unit Distributions	Transformation	References
A new two-parameter unit Bilal distribution	$Y = \exp(-X/\alpha)$	[8]
The unit-transmuted Lindley distribution	$Y = X/(X+1)$	[9]
The unit two parameters Mirra distribution	$Y = 1/(X+1)$	[10]
Unit Haq distribution	$Y = \exp(-X)$	[11]
The unit new X-Lindley distribution	$Y = \exp(-X)$	[12]
Laplace-logistic unit distribution	$Y = (1+\exp(X))^{-\frac{1}{\alpha}}$	[13]
Cauchy–logistic unit distribution	$Y = (1+\exp(X))^{-\frac{1}{\alpha}}$	[14]
Gumbel–logistic unit distribution	$Y = (1+\exp(X))^{-\frac{1}{\alpha}}$	[15]
The truncated inverted arctan power distribution	$Y = \frac{\arctan(\alpha X)}{\arctan(\alpha)}$	[16]
The unit Zeghdoudi distribution	$Y = X/(X+1)$	[17]
Unit Maxwell-Boltzmann distribution	$Y = \exp(-X)$	[18]
The unit inverse Weibull distribution	$Y = \exp(-X)$	[19]
Unit modified Burr-III distribution	$Y = X/(X+1)$	[20]
Unit exponential Pareto distribution	$Y = X/(1-X)$	[21]
Poisson-unit-Weibull distribution	$Y = \exp(-X)$	[22]
Unit Xgamma distribution	$Y = X/(X+1)$	[23]
The unit Muth distribution	$Y = \exp(-X)$	[24]
Unit-exponentiated half-logistic distribution	$Y = \exp(-X)$	[25]
The unit Teissier distribution	$Y = \exp(-X)$	[26]
Unit half-logistic geometric distribution	$Y = \exp(-X)$	[27]
The unit log–log distribution	$Y = \exp(-X)$	[28]
Unit Zeghdoudi distribution	$Y = X/(X+1)$	[29]
The unit inverse Lindley distribution	$Y = X/(X+1)$	[30]

In this paper, a new alternative unit distribution is introduced based on the two-parameter Aradhana distribution [31]. The rest of the paper is organized as follows: In Section 2, a new unit distribution is introduced and several statistical properties are derived, including the hazard rate function (hrf), survival function (sf), moments, skewness (S), kurtosis (K) coefficients, and order statistics. In Section 3, focuses on parameter estimation for the proposed distribution using methods such as maximum likelihood estimation (MLE), Cramér-von Mises estimation (CvME) Anderson-Darling estimation (ADE), least squares estimation (LSE), weighted least squares estimation (WLSE), maximum product of spacings estimation (MPSE), minimum spacing absolute distance estimation (MSADE), minimum spacing absolute log distance estimation (MSALDE) Anderson-Darling left tail estimation (ADLTE) Anderson-Darling left tail second-order estimation (ADLTSOE) Anderson-Darling right tail estimation (ADRTE) and Kolmogorov method estimation (KME). In Section 4, a Monte Carlo simulation study is conducted to compare the estimators based on some criteria across various scenarios. In Section 5, two real-data applications are presented in which the proposed distribution is compared with well-known competitors such as B, KW, unit-Lindley (UL), unit Teisser (UT), UW, and unit Burr-XII (UB-XII), thereby demonstrating its superiority. Section 6 concludes with a discussion of the findings and overall interpretations for the reader.

2. THE BOUNDED VERSION OF THE TWO-PARAMETER ARADHANA DISTRIBUTION

Let the random variable Y follow the two-parameter Aradhana distribution (AD), with the cumulative distribution function (cdf) and probability density function (pdf) defined, respectively, as

$$F_{AD}(y;\boldsymbol{\psi}) = 1 - \left[1 + \frac{\theta y(\theta y + 2\theta\alpha + 2)}{\theta^2\alpha^2 + 2\theta\alpha + 2}\right]\exp(-\theta y), \;\; y > 0 \tag{1}$$

and

$$f_{AD}(y;\boldsymbol{\psi}) = \frac{\theta^3}{\theta^2\alpha^2 + 2\theta\alpha + 2}(\alpha + y)^2 \exp(-\theta y), \;\; y > 0, \tag{2}$$

where $\theta > 0,\ \alpha > 0$ and $\boldsymbol{\psi} = (\alpha, \theta)$. If the random variable Y has the pdf provided in Equation (2), then the cdf and pdf of the random variables $X = \exp(-Y)$ can be derived, respectively, as

$$F_{UTPA}(x;\boldsymbol{\psi}) = \left(1 - \frac{\theta\log(x)(-\theta\log(x) + 2\theta\alpha + 2)}{\theta^2\alpha^2 + 2\theta\alpha + 2}\right)x^{\theta}, \tag{3}$$

and

$$f_{UTPA}(x;\boldsymbol{\psi}) = \frac{\theta^3(\alpha - \log(x))^2 x^{\theta-1}}{(\theta^2\alpha^2 + 2\theta\alpha + 2)}, \tag{4}$$

where $0 < x < 1$ and $\theta > 0, \alpha > 0$ are the model parameters. The new distribution, whose cdf is given in Equation (3) and pdf in Equation (4) is referred to as the unit two-parameter Aradhana distribution (UTPA), and it is denoted by $UTPA(\boldsymbol{\psi})$. Moreover, the sf of the UTPA distribution can also be easily obtained as $S_{UTPA}(x;\boldsymbol{\psi}) = 1 - F_{UTPA}(x;\boldsymbol{\psi})$. The hrf of the UTPA distribution can be expressed as

$$h_{UTPA}(x;\boldsymbol{\psi}) = \frac{\theta^3(\alpha - \log(x))^2 x^{\theta-1}}{(\theta^2\alpha^2 + 2\theta\alpha + 2)\left(1 - \left(1 - \frac{\theta\log(x)(-\theta\log(x) + 2\theta\alpha + 2)}{\theta^2\alpha^2 + 2\theta\alpha + 2}\right)x^{\theta}\right)}.$$

Figs 1 and 2 present the pdf and hrf of the UTPA distribution for different parameter values. As seen in Figure 1, the pdf and hrf exhibit diverse patterns. In particular, the pdf can take on decreasing, increasing and unimodal forms, whereas the hrf demonstrates decreasing, increasing and bathtub behaviors.

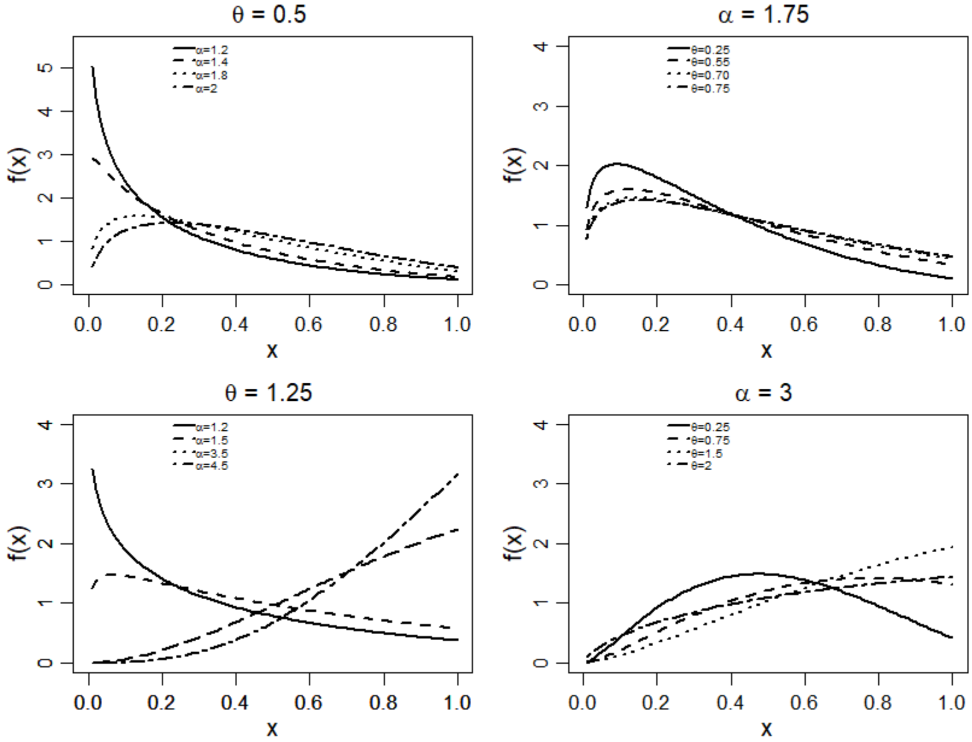

Figure 1. The pdf of UTPA distribution for various parameter choices.

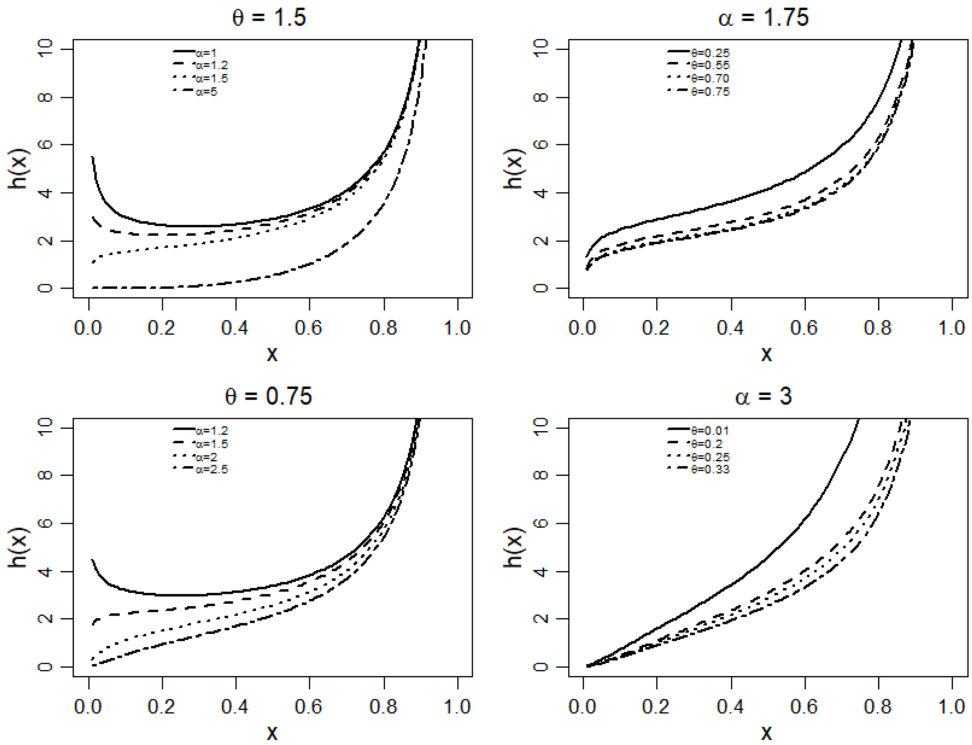

Figure 2. The hrf of UTPA distribution for various parameter choices.

2.1. MOMENTS

The *r*-th moment of the UTPA distribution obtained as

$$\begin{aligned} E\left(X^r\right) &= \int_0^1 x^r f\left(x\right) dx \\ &= \int_0^1 \frac{\theta^3 \left(\alpha - \log\left(x\right)\right)^2 x^{\theta-1+r}}{\left(\theta^2\alpha^2 + 2\theta\alpha + 2\right)} dx \\ &= \frac{\theta^3 \left(\theta^2\alpha^2 + \left(2\alpha + 2\alpha^2 r\right)\theta + \alpha^2 r^2 + 2 + 2\alpha r\right)}{\left(\theta^2\alpha^2 + 2\theta\alpha + 2\right)\left(\theta + r\right)^3}. \end{aligned} \tag{5}$$

The first four moments and the variance, skewness and kurtosis of the proposed distribution can be easily derived by substituting $r = 1, 2, 3$ and 4 into the general moment expression given in Equation (5). The Skewness (S) and Kurtosis (K) of the UTPA distribution can be computed, respectively, as follows:

$$S = \frac{E\left(X^3\right) - 3E\left(X\right)E\left(X^2\right) + 2\left\{E\left(X\right)\right\}^3}{\left\{V\left(X\right)\right\}^{3/2}} \tag{6}$$

and

$$K = \frac{E\left(X^4\right) - 4E\left(X\right)E\left(X^3\right) + 6E\left(X^2\right)\left\{E\left(X\right)\right\}^2 - 3\left\{E\left(X\right)\right\}^4}{\left\{V\left(X\right)\right\}^2}. \tag{7}$$

The expected value, variance, S and K are provided in Table 2 and Figure 3 for selected values of α and θ. According to Table 2 and Fig. 3, as the α and θ parameters increase, the expected value monotonically increases.

Table 2. Expected value, variances, S and K for various choices of θ and α .

θ	α	$E(X)$	$V(X)$	S	K
0.5	0.5	0.0587	0.0155	3.5376	17.7684
	1.5	0.1054	0.0353	2.4770	9.0091
	3	0.1596	0.0553	1.7675	5.2776
	5	0.2048	0.0687	1.3730	3.8071
2	0.5	0.4296	0.0654	0.3575	2.1326
	1.5	0.5447	0.0710	-00957	1.9095
	3	0.5985	0.0668	-0.3097	2.0389
	5	0.6242	0.0633	-0.4095	2.1518
3.5	0.5	0.6354	0.0496	-0.3204	2.2167
	1.5	0.7171	0.0425	-0.6912	2.7054
	3	0.7459	0.0378	-0.8272	3.0148
	5	0.7584	0.0354	-0.8837	3.1664
5	0.5	0.7425	0.0332	-0.6880	2.8101
	1.5	0.7979	0.0262	-0.9998	3.5556
	3	0.8151	0.0233	-1.0956	3.8699
	5	0.8223	0.0220	-1.1328	4.0044

On the other hand, while the variance increases for small values of θ, it decreases as θ grows. Regarding S, the distribution is right-skewed for small values of α and θ. However, as θ and α increase, the distribution shifts to the left and becomes left-skewed. For K, it is generally lower for small values of α and θ, but as α and θ increase, kurtosis also increases.

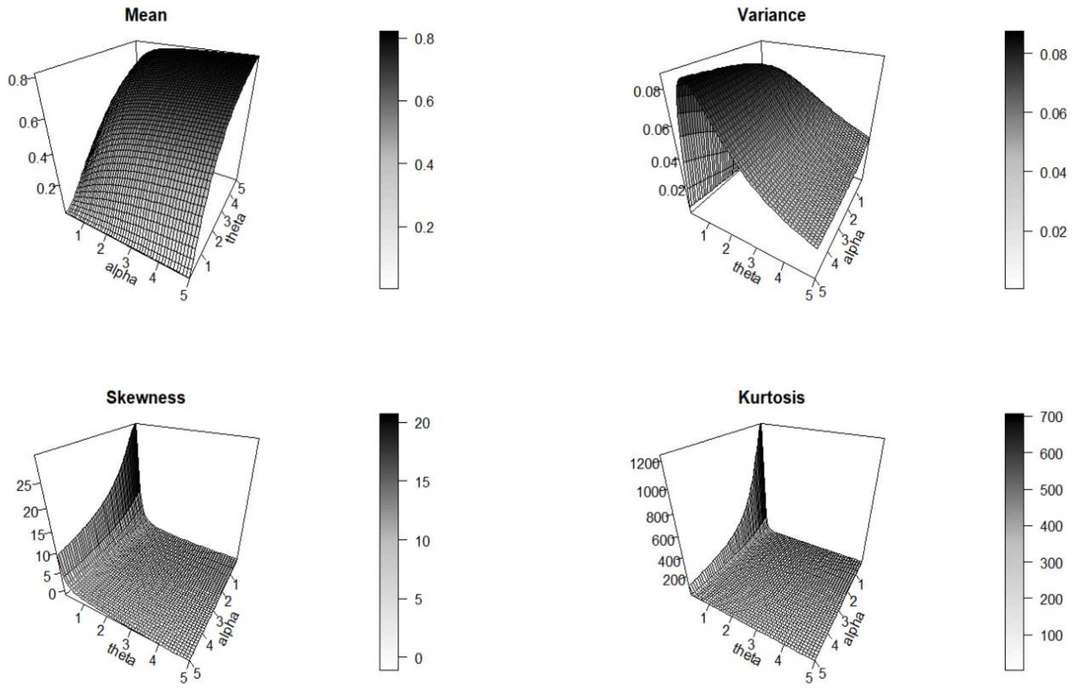

Figure 3. The mean, variance, S, and K of UTPA distribution for various parameter choices.

2.2. ORDER STATISTICS

In this subsection, we explore results related to the order statistics of the UTPA distribution. Let $X_1, X_2, \dots, X_n$ be a random sample from the UTPA distribution and $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ indicates the corresponding order statistics. The cdf and pdf of the $X_{(r)}$ are presented for general form, respectively, by

$$F_{X_{(r)}}(x;\boldsymbol{\psi}) = \sum_{i=r}^{n} \binom{n}{i} F(x;\boldsymbol{\psi})^{i} \left\{1 - F(x;\boldsymbol{\psi})\right\}^{n-i}$$

$$= \sum_{i=r}^{n} \sum_{j=0}^{n-i} (-1)^{j} \binom{n}{i} \binom{n-i}{j} F(x;\boldsymbol{\psi})^{i+j},$$

and

$$f_{X_{(r)}}(x;\boldsymbol{\psi}) = \frac{1}{B(r,n-r+1)} F(x;\boldsymbol{\psi})^{r-1} \left\{1 - F(x;\boldsymbol{\psi})\right\}^{n-r} f(x;\boldsymbol{\psi})$$

$$= \frac{1}{B(r,n-r+1)} \sum_{i=0}^{n-r} (-1)^{i} \binom{n-r}{i} F(x;\boldsymbol{\psi})^{r+i-1} f(x;\boldsymbol{\psi}),$$

where $r = 1, 2, \ldots, n$ and $B(.;.)$ is the classical B function. The cdf and pdf of the $X_{(r)}$ order statistic of the UTPA distribution is also derived by

$$F_{X_{(r)}}(x;\boldsymbol{\psi}) = \sum_{i=r}^{n}\sum_{j=0}^{n-i}(-1)^j \binom{n}{i}\binom{n-i}{j} \times \left\{\left(1 - \frac{\theta\log(x)\left(-\theta\log(x) + 2\theta\alpha + 2\right)}{\theta^2\alpha^2 + 2\theta\alpha + 2}\right)x^{\theta}\right\}^{i+j},$$

and

$$f_{X_{(r)}}(x;\boldsymbol{\psi}) = \frac{\theta^3\left(\alpha - \log(x)\right)^2 x^{\theta-1}}{\left(\theta^2\alpha^2 + 2\theta\alpha + 2\right)B(r, n-r+1)} \times \sum_{i=0}^{n-r}(-1)^i\binom{n-r}{i}\left\{\left(1 - \frac{\theta\log(x)\left(-\theta\log(x) + 2\theta\alpha + 2\right)}{\theta^2\alpha^2 + 2\theta\alpha + 2}\right)x^{\theta}\right\}^{r+i-1}.$$

When $r = 1$ and $r = n$, the cdf and pdf of $X_{(1)} = \min\{X_1, X_2, \ldots, X_n\}$ and $X_{(n)} = \max\{X_1, X_2, \ldots, X_n\}$ are obtained.

2.3. STOCHASTIC ORDERING

In this subsection, stochastic ordering properties of the UTPA distribution are discussed. Stochastic ordering finds applications in various fields such as actuarial science, reliability, insurance, and economics [32].

Theorem 1. Let the random variable $X \sim UTPA(\alpha, \theta_1)$ with pdf $f_x(x)$ and the random variable $Y \sim UTPA(\alpha, \theta_2)$ with the pdf $f_y(x)$. Then, if $\theta_1 < \theta_2$, X is smaller than Y in the likelihood ratio order, the ratio function of the corresponding pdf defined by $g(x) = f_x(x) / f_y(x)$ is decreasing in x.

Proof: After performing some algebraic manipulations, we obtain

$$g(x) = \frac{f_x(x)}{f_y(x)} = \frac{\theta_1^3 x^{(\theta_1 - 1)}\left(\theta_2^2\alpha^2 + 2\theta_2\alpha + 2\right)}{\theta_2^3 x^{(\theta_2 - 1)}\left(\theta_1^2\alpha^2 + 2\theta_1\alpha + 2\right)}$$

Upon differentiation with respect to *x,* and after some algebraic manipulations, we obtain

$$g'(x) = \frac{d\log\left[g(x)\right]}{dx} = \frac{\theta_1 - \theta_2}{x}.$$

It is clear that for $\theta_1 < \theta_2$, $g'(x) < 0$, indicating that $g(x)$ is decreasing function of x. Thus, the desired likelihood ratio order is achieved.

Corollary 1. It follows from [32] that, under the conditions outlined in Theorem 1, X is smaller than Y with respect to the hazard rate, mean residual life, and stochastic orders.

The stochastic ordering result indicates that the distribution corresponding to different parameter values can be systematically ordered. Since likelihood ratio ordering implies hazard rate and usual stochastic ordering, the proposed model enables comparison of reliability behavior across parameter configurations. The direction of the ordering depends on the parameter setting, and therefore provides a mechanism for ranking systems according to their relative risk characteristics.

2.4. DATA GENERATION FOR UTPA DISTRIBUTION

This subsection presents an acceptance-rejection (AR) algorithm for generating random data from the UTPA distribution. For a detailed explanation of the AR algorithm, please refer to [33]. The steps involved in the AR algorithm for generating data from the UTPA distribution are outlined in Algorithm 1.

Algorithm 1

Step 1. Generating a sample random variable Y from the standard uniform distribution with the following pdf

$$h(y) = 1,\ 0 < y < 1.$$

Step 2. A sample is generated from a uniform distribution independent of Y, say $U \sim U(0,1)$.

Step 3. If

$$U < \frac{f(Y)}{k \times h(Y)}$$

then $Y = X$ (Accept), otherwise go to **Step 1** (Reject), where f is the pdf in Equation (3) and

$$k = \max_{y \in R^+} \frac{f(y)}{h(y)}$$

The output of Algorithm 1 provides a sample for the variable Y from the UTPA distribution.

2.5. MODE

The mode is the most frequently occurring value in a data set. For the UTPA distribution, the corresponding mode can be determined using Equation $f'(x) = 0$. That is obtained by,

$$\frac{\theta^3\left(\log(x)-\alpha\right)x^{(\theta-2)}\left(2-\theta\alpha+\theta\log(x)+\alpha-\log(x)\right)}{\theta^2\alpha^2+2\theta\alpha+2}=0$$

where, $x\in(0,1)$ and $\alpha>0,\theta>0$.

2.6. QUANTILE FUNCTION

It may be noted that $F(x)$ in Equation (3) is continuous and strictly increasing, so the quantile function of X is defined as follows:

$$Q_X(u)=x_u=F_X^{-1}(u),\ u\in[0,1].$$

3. POINT ESTIMATION

In this section, twelve different estimators are considered for estimating the unknown parameters of the UTPA distribution. These estimators are as follows: MLE, ADE [34], CvME [35], LSE [36], WLSE [36], KME, MPSE [37], MSADE, MSALDE, ADLTE, ADLTSOE and ADRTE. The performance of these estimators is evaluated through a Monte Carlo simulation.

Let $X_1,X_2\ldots,X_n$ represent a random sample from the UTPA distribution and $x_1,x_2\ldots,x_n$ denote the observed values of the sample. The order statistics based on the sample $x_1,x_2\ldots,x_n$ are represented as $x_{(1)},x_{(2)}\ldots,x_{(n)}$.

We first consider the MLE method, which is widely used due to its desirable asymptotic properties. The likelihood and log-likelihood (ℓ) functions are expressed, respectively, by

$$L(\boldsymbol{\psi})=\prod_{i=1}^{n}\left\{\frac{\theta^3\left(\alpha-\log(x_i)\right)^2x_i^{\theta-1}}{\left(\theta^2\alpha^2+2\theta\alpha+2\right)}\right\}$$

and

$$\ell(\boldsymbol{\psi})=3n\log(\theta)+2\sum_{i=1}^{n}\log\left[\alpha-\log(x_i)\right]+(\theta-1)\sum_{i=1}^{n}\log(x_i)-n\log\left(\theta^2\alpha^2+2\theta\alpha+2\right)$$

Alternative estimators are also considered to evaluate the robustness and comparative performance of the proposed distribution. Now, the sum function to obtain the other estimators is defined as follows:

$$ADE(\boldsymbol{\psi})=-n-\sum_{i=1}^{n}\left(\frac{2i-1}{n}\right)\left\{\log\left[\left(1-\frac{\theta\log\left(x_{(i)}\right)\left(-\theta\log\left(x_{(i)}\right)+2\theta\alpha+2\right)}{\theta^2\alpha^2+2\theta\alpha+2}\right)e^{\left(\theta\log\left(x_{(i)}\right)\right)}\right]\right. \quad (8)$$

$$+\left[1-\left(1-\frac{\theta\log\left(x_{(n+1-i)}\right)\left(-\theta\log\left(x_{(n+1-i)}\right)+2\theta\alpha+2\right)}{\theta^2\alpha^2+2\theta\alpha+2}\right)e^{\left(\theta\log\left(x_{(n+1-i)}\right)\right)}\right]\Bigg\}\ ,$$

$$CvME(\psi)=\frac{1}{12n}+\sum_{i=1}^{n}\left[\left(1-\frac{\theta\log\left(x_{(i)}\right)\left(-\theta\log\left(x_{(i)}\right)+2\theta\alpha+2\right)}{\theta^2\alpha^2+2\theta\alpha+2}\right)e^{\left(\theta\log\left(x_{(i)}\right)\right)}-\frac{2i-1}{2n}\right]^2\ , \tag{9}$$

$$LSE(\psi)=\sum_{i=1}^{n}\left[\left(1-\frac{\theta\log\left(x_{(i)}\right)\left(-\theta\log\left(x_{(i)}\right)+2\theta\alpha+2\right)}{\theta^2\alpha^2+2\theta\alpha+2}\right)e^{\left(\theta\log\left(x_{(i)}\right)\right)}-\frac{i}{n+1}\right]^2\ , \tag{10}$$

$$WLSE(\psi)=\left\{\sum_{i=1}^{n}\left[\frac{(n+1)^2(n+2)}{i(n-i+1)}\right]\right.$$
$$\left.\left[\left(1-\frac{\theta\log\left(x_{(i)}\right)\left(-\theta\log\left(x_{(i)}\right)+2\theta\alpha+2\right)}{\theta^2\alpha^2+2\theta\alpha+2}\right)e^{\left(\theta\log\left(x_{(i)}\right)\right)}-\frac{i}{n+1}\right]^2\right\}\ , \tag{11}$$

$$KME(\psi)=\max_{1\le i\le n}\left\{\frac{i}{n}-\left(1-\frac{\theta\log\left(x_{(i)}\right)\left(-\theta\log\left(x_{(i)}\right)+2\theta\alpha+2\right)}{\theta^2\alpha^2+2\theta\alpha+2}\right)e^{\left(\theta\log\left(x_{(i)}\right)\right)}\right.$$
$$\left.,\left(1-\frac{\theta\log\left(x_{(i)}\right)\left(-\theta\log\left(x_{(i)}\right)+2\theta\alpha+2\right)}{\theta^2\alpha^2+2\theta\alpha+2}\right)e^{\left(\theta\log\left(x_{(i)}\right)\right)}-\frac{i-1}{n}\right\}, \tag{12}$$

$$MPSE(\psi)=\frac{1}{n+1}+\left\{\sum_{i=1}^{n+1}\log\left(1-\frac{\theta\log\left(x_{(i-1)}\right)\left(-\theta\log\left(x_{(i-1)}\right)+2\theta\alpha+2\right)}{\theta^2\alpha^2+2\theta\alpha+2}\right)e^{\left(\theta\log\left(x_{(i-1)}\right)\right)}\right.$$
$$\left.-\left(1-\frac{\theta\log\left(x_{(i)}\right)\left(-\theta\log\left(x_{(i)}\right)+2\theta\alpha+2\right)}{\theta^2\alpha^2+2\theta\alpha+2}\right)e^{\left(\theta\log\left(x_{(i)}\right)\right)}\right\}\ , \tag{13}$$

$$MSADE(\psi)=\left\{\sum_{i=1}^{n+1}\log\left(1-\frac{\theta\log\left(x_{(i-1)}\right)\left(-\theta\log\left(x_{(i-1)}\right)+2\theta\alpha+2\right)}{\theta^2\alpha^2+2\theta\alpha+2}\right)e^{\left(\theta\log\left(x_{(i-1)}\right)\right)}\right. \tag{14}$$

$$-\left(1-\frac{\theta\log\left(x_{(i)}\right)\left(-\theta\log\left(x_{(i)}\right)+2\theta\alpha+2\right)}{\theta^2\alpha^2+2\theta\alpha+2}\right)e^{\left(\theta\log\left(x_{(i)}\right)\right)}-\frac{1}{n+1}\Bigg\} \quad ,$$

$$MSALDE(\psi)=\left\{\sum_{i=1}^{n+1}\log\left(1-\frac{\theta\log\left(x_{(i-1)}\right)\left(-\theta\log\left(x_{(i-1)}\right)+2\theta\alpha+2\right)}{\theta^2\alpha^2+2\theta\alpha+2}\right)e^{\left(\theta\log\left(x_{(i-1)}\right)\right)}\right.$$
$$\left.-\left(1-\frac{\theta\log\left(x_{(i)}\right)\left(-\theta\log\left(x_{(i)}\right)+2\theta\alpha+2\right)}{\theta^2\alpha^2+2\theta\alpha+2}\right)e^{\left(\theta\log\left(x_{(i)}\right)\right)}-\log(n+1)\right\} \quad , \tag{15}$$

$$ADLTE(\psi)=-\frac{3n}{2}-2\sum_{i=1}^{n}\left[\left(1-\frac{\theta\log\left(x_{(i)}\right)\left(-\theta\log\left(x_{(i)}\right)+2\theta\alpha+2\right)}{\theta^2\alpha^2+2\theta\alpha+2}\right)e^{\left(\theta\log\left(x_{(i)}\right)\right)}\right]$$
$$-\left(\frac{2i-1}{n}\right)\sum_{i=1}^{n}\left[\left(1-\frac{\theta\log\left(x_{(i)}\right)\left(-\theta\log\left(x_{(i)}\right)+2\theta\alpha+2\right)}{\theta^2\alpha^2+2\theta\alpha+2}\right)e^{\left(\theta\log\left(x_{(i)}\right)\right)}\right] \quad , \tag{16}$$

$$ADLTSOE(\psi)=2\sum_{i=1}^{n}\left[\left(1-\frac{\theta\log\left(x_{(i)}\right)\left(-\theta\log\left(x_{(i)}\right)+2\theta\alpha+2\right)}{\theta^2\alpha^2+2\theta\alpha+2}\right)e^{\left(\theta\log\left(x_{(i)}\right)\right)}\right]$$
$$+\left(\frac{2i-1}{n}\right)\sum_{i=1}^{n}\left[\left(1-\frac{\theta\log\left(x_{(i)}\right)\left(-\theta\log\left(x_{(i)}\right)+2\theta\alpha+2\right)}{\theta^2\alpha^2+2\theta\alpha+2}\right)e^{\left(\theta\log\left(x_{(i)}\right)\right)}\right]^{-1} \quad , \tag{17}$$

and

$$ADRTE(\psi)=\frac{n}{2}-2\sum_{i=1}^{n}\left[\left(1-\frac{\theta\log\left(x_{(i)}\right)\left(-\theta\log\left(x_{(i)}\right)+2\theta\alpha+2\right)}{\theta^2\alpha^2+2\theta\alpha+2}\right)e^{\left(\theta\log\left(x_{(i)}\right)\right)}\right]$$
$$-\left(\frac{2i-1}{n}\right)\sum_{i=1}^{n}\left\{1-\left[\left(1-\frac{\theta\log\left(x_{(n+1-i)}\right)\left(-\theta\log\left(x_{(n+1-i)}\right)+2\theta\alpha+2\right)}{\theta^2\alpha^2+2\theta\alpha+2}\right)e^{\left(\theta\log\left(x_{((n+1-i))}\right)\right)}\right]\right\}. \tag{18}$$

Then, for twelve estimators, the $\boldsymbol{\psi}$ estimators are obtained using optimization methods such as Nelder-Mead and BFGS, which handle both maximization and minimization while addressing all optimization problems. Then, the estimators are derived using Equations (8) – (18), respectively, as

$$\boldsymbol{\psi}_1 = \arg\max_{\psi} MLE(\boldsymbol{\psi})$$

$$\boldsymbol{\psi}_2 = \arg\min_{\psi} ADE(\boldsymbol{\psi})$$

$$\boldsymbol{\psi}_3 = \arg\min_{\psi} CvME(\boldsymbol{\psi})$$

$$\boldsymbol{\psi}_4 = \arg\min_{\psi} LSE(\boldsymbol{\psi})$$

$$\boldsymbol{\psi}_5 = \arg\min_{\psi} WLSE(\boldsymbol{\psi})$$

$$\boldsymbol{\psi}_6 = \arg\min_{\psi} KME(\boldsymbol{\psi})$$

$$\boldsymbol{\psi}_7 = \arg\max_{\psi} MPSE(\boldsymbol{\psi})$$

$$\boldsymbol{\psi}_8 = \arg\min_{\psi} MSADE(\boldsymbol{\psi})$$

$$\boldsymbol{\psi}_9 = \arg\min_{\psi} MSALDE(\boldsymbol{\psi})$$

$$\boldsymbol{\psi}_{10} = \arg\min_{\psi} ADLTE(\boldsymbol{\psi})$$

$$\boldsymbol{\psi}_{11} = \arg\min_{\psi} ADLTSOE(\boldsymbol{\psi})$$

$$\boldsymbol{\psi}_{12} = \arg\min_{\psi} ADRTE(\boldsymbol{\psi})$$

In the following section, a simulation study is conducted to evaluate the performance of these estimators.

4. MONTE CARLO SIMULATION STUDY

In this subsection, a Monte Carlo simulation study with 10000 replications is conducted to evaluate the performance of the MLE, ADE, CvME, LSE, WLSE, KME, MPSE, MSADE, MSALDE, ADLTE, ADLTSOE, and ADRTE across different sample sizes. Four different scenarios are considered for the simulation study. The scenarios are defined as follows: $\boldsymbol{\psi}_{S1} = (0.4, 0.9)$, $\boldsymbol{\psi}_{S2} = (0.7, 1.6)$, $\boldsymbol{\psi}_{S3} = (2.7, 1.4)$ and $\boldsymbol{\psi}_{S4} = (1.5, 0.3)$. The random data generation in the simulation study is conducted using Algorithm 1. Generally, a mean-based approach is used to evaluate estimation performance. However, when parameter estimates deviate significantly from the true values, this approach may be undesirable and a median-based approach can be considered. The median-based approach provides estimates that are not influenced by outliers, whereas the mean-based approach is sensitive to them [38]. Therefore, to obtain more meaningful results, median $(\theta_i - \theta)^2$, median $|\theta_i - \theta|$ and median $|\theta_i - \theta| / \theta$ are used for based on median square error (MSE), absolute average bias (ABB) and mean root error (MRE), respectively. These median-based formulas are used to compute the MSE, ABB, and MRE for the parameters. Tables 3–6 present the results of 12 different estimators in terms of MSE, ABB, and MRE for various sample sizes across four scenarios. For each sample size and parameter value, a ranking is performed based on MSE, ABB, and MRE. In this ranking system, $^{(1)}$ represents the best estimator, while $^{(12)}$ represents the worst. A comprehensive ranking table summarizing these results is provided in Table 7. Table 7 offers a comparative evaluation of the twelve different estimators across four scenarios and varying sample sizes. According to Table 7, KME consistently outperformed all other estimators, achieving the best overall ranking, highlighting its robustness across various conditions. Similarly, MLE, MSADE and ADE are ranked second, third and fourth, respectively, demonstrating strong performance. In contrast, ADLTE, LSE and ADLTSOE exhibit weak performance, indicating their limited applicability, poor estimation accuracy and suboptimal nature. MPSE, MSALDE, WLSE, CvME and ADRTE are ranked in the mid-range, suggesting moderate effectiveness. This result indicates that the KME method is the most successful estimator in terms of MSE, ABB, and MRE. In contrast, the LSE and ADLTSOE methods, having the highest overall rank values, exhibit the weakest overall performance. When the evaluation is conducted with respect to sample size, it is observed that, particularly in small-sample cases such as n=25, the KME method generally ranks first across all scenarios. This finding indicates that the KME method maintains low MSE, ABB, and MRE even under small-sample conditions, thereby demonstrating strong and stable performance. In contrast, the MLE method ranks second in most cases and emerges as a reliable alternative to KME. For sample sizes n=50, 75, and 100, a decreasing trend in MSE, ABB, and MRE values is observed for all methods as the sample size increases. However, the superiority ranking among the methods remains largely unchanged, with KME delivering the best performance in most scenarios. For sample sizes n=250 and n=500, the error measures decrease substantially, and the differences among the methods narrow. Nevertheless, even under these conditions, the KME method maintains its overall superiority. Although in certain scenarios the MLE method performs very close to KME, the overall ranking clearly indicates that KME remains the most successful method. Overall, all estimators showed consistent performance across different sample sizes and scenarios, maintaining stable rankings in the results. In conclusion, KME, MLE, MSADE, and ADE have emerged as the most effective estimators across different scenarios and sample sizes, whereas ADLTE, LSE, and ADLTSOE have proven to be weak estimators, making them unsuitable for estimation purposes in the given scenarios.

Table 3. The MSE, ABB and MRE values for ψ_{S1}.

n		Parameter	MLE	ADE	CvME	LSE	WLSE	KME	MPSE	MSADE	MSALDE	ADLTE	ADLTSOE	ADRTE
25	MSE	α	0.1150$^{(5)}$	0.1053$^{(3)}$	0.1579$^{(9)}$	0.1744$^{(10)}$	0.1361$^{(6)}$	0.0047$^{(1)}$	0.1488$^{(7)}$	0.1041$^{(2)}$	0.1542$^{(8)}$	0.1850$^{(11)}$	0.5663$^{(12)}$	0.1115$^{(4)}$
		θ	0.0111$^{(2)}$	0.0121$^{(3)}$	0.0165$^{(10)}$	0.0169$^{(11)}$	0.0144$^{(5)}$	0.0067$^{(1)}$	0.0145$^{(6)}$	0.0146$^{(7)}$	0.0159$^{(9)}$	0.0159$^{(8)}$	0.0230$^{(12)}$	0.0137$^{(4)}$
	ABB	α	0.3392$^{(5)}$	0.3244$^{(3)}$	0.3974$^{(9)}$	0.4176$^{(10)}$	0.3689$^{(6)}$	0.0688$^{(1)}$	0.3857$^{(7)}$	0.3226$^{(2)}$	0.3926$^{(8)}$	0.4302$^{(11)}$	0.7525$^{(12)}$	0.3339$^{(4)}$
		θ	0.1054$^{(2)}$	0.1099$^{(3)}$	0.1283$^{(10)}$	0.1302$^{(11)}$	0.1199$^{(5)}$	0.0819$^{(1)}$	0.1204$^{(6)}$	0.1209$^{(7)}$	0.1260$^{(9)}$	0.1260$^{(8)}$	0.1516$^{(12)}$	0.1169$^{(4)}$
	MRE	α	0.8480$^{(5)}$	0.8111$^{(3)}$	0.9935$^{(9)}$	1.0441$^{(10)}$	0.9222$^{(6)}$	0.1720$^{(1)}$	0.9643$^{(7)}$	0.8065$^{(2)}$	0.9816$^{(8)}$	1.0754$^{(11)}$	1.8813$^{(12)}$	0.8347$^{(4)}$
		θ	0.1171$^{(2)}$	0.1222$^{(3)}$	0.1425$^{(10)}$	0.1446$^{(11)}$	0.1332$^{(5)}$	0.0910$^{(1)}$	0.1338$^{(6)}$	0.1343$^{(7)}$	0.1400$^{(9)}$	0.1400$^{(8)}$	0.1684$^{(12)}$	0.1299$^{(4)}$
	Sum Rank		21$^{(3)}$	18$^{(2)}$	57$^{(9.5)}$	63$^{(11)}$	33$^{(6)}$	6$^{(1)}$	39$^{(7)}$	27$^{(5)}$	51$^{(8)}$	57$^{(9.5)}$	72$^{(12)}$	24$^{(4)}$
50	MSE	α	0.0767$^{(4)}$	0.0786$^{(5)}$	0.1075$^{(9)}$	0.1158$^{(10)}$	0.0907$^{(6)}$	0.0042$^{(1)}$	0.0940$^{(7)}$	0.0741$^{(2)}$	0.1005$^{(8)}$	0.1262$^{(11)}$	0.4023$^{(12)}$	0.0759$^{(3)}$
		θ	0.0078$^{(2)}$	0.0087$^{(3)}$	0.0112$^{(10)}$	0.0113$^{(11)}$	0.0097$^{(5)}$	0.0045$^{(1)}$	0.0093$^{(4)}$	0.0107$^{(7)}$	0.0108$^{(9)}$	0.0108$^{(8)}$	0.0188$^{(12)}$	0.0097$^{(6)}$
	ABB	α	0.2770$^{(4)}$	0.2804$^{(5)}$	0.3278$^{(9)}$	0.3402$^{(10)}$	0.3012$^{(6)}$	0.0648$^{(1)}$	0.3066$^{(7)}$	0.2723$^{(2)}$	0.3170$^{(8)}$	0.3552$^{(11)}$	0.6343$^{(12)}$	0.2754$^{(3)}$
		θ	0.0881$^{(2)}$	0.0931$^{(3)}$	0.1056$^{(10)}$	0.1061$^{(11)}$	0.0984$^{(5)}$	0.0674$^{(1)}$	0.0963$^{(4)}$	0.1035$^{(7)}$	0.1041$^{(9)}$	0.1038$^{(8)}$	0.1371$^{(12)}$	0.0987$^{(6)}$
	MRE	α	0.6925$^{(4)}$	0.7011$^{(5)}$	0.8196$^{(9)}$	0.8506$^{(10)}$	0.7530$^{(6)}$	0.1619$^{(1)}$	0.7665$^{(7)}$	0.6807$^{(2)}$	0.7925$^{(8)}$	0.8881$^{(11)}$	1.5856$^{(12)}$	0.6886$^{(3)}$
		θ	0.0978$^{(2)}$	0.1035$^{(3)}$	0.1173$^{(10)}$	0.1179$^{(11)}$	0.1094$^{(5)}$	0.0749$^{(1)}$	0.1070$^{(4)}$	0.1150$^{(7)}$	0.1157$^{(9)}$	0.1153$^{(8)}$	0.1523$^{(12)}$	0.1096$^{(6)}$
	Sum Rank		18$^{(2)}$	24$^{(3)}$	57$^{(9.5)}$	63$^{(11)}$	33$^{(6.5)}$	6$^{(1)}$	33$^{(6.5)}$	27$^{(4.5)}$	51$^{(8)}$	57$^{(9.5)}$	72$^{(12)}$	27$^{(4.5)}$
75	MSE	α	0.0565$^{(2)}$	0.0597$^{(5)}$	0.0818$^{(9)}$	0.0879$^{(10)}$	0.0667$^{(6)}$	0.0038$^{(1)}$	0.0667$^{(7)}$	0.0579$^{(3)}$	0.0740$^{(8)}$	0.0978$^{(11)}$	0.2985$^{(12)}$	0.0588$^{(4)}$
		θ	0.0060$^{(2)}$	0.0066$^{(3)}$	0.0083$^{(8)}$	0.0084$^{(10)}$	0.0072$^{(5)}$	0.0035$^{(1)}$	0.0070$^{(4)}$	0.0084$^{(11)}$	0.0083$^{(9)}$	0.0082$^{(7)}$	0.0153$^{(12)}$	0.0076$^{(6)}$
	ABB	α	0.2377$^{(2)}$	0.2443$^{(5)}$	0.2860$^{(9)}$	0.2965$^{(10)}$	0.2582$^{(6)}$	0.0616$^{(1)}$	0.2584$^{(7)}$	0.2407$^{(3)}$	0.2721$^{(8)}$	0.3128$^{(11)}$	0.5463$^{(12)}$	0.2424$^{(4)}$
		θ	0.0772$^{(2)}$	0.0815$^{(3)}$	0.0911$^{(8)}$	0.0917$^{(10)}$	0.0847$^{(5)}$	0.0595$^{(1)}$	0.0836$^{(4)}$	0.0918$^{(11)}$	0.0913$^{(9)}$	0.0906$^{(7)}$	0.1236$^{(12)}$	0.0870$^{(6)}$
	MRE	α	0.5943$^{(2)}$	0.6108$^{(5)}$	0.7149$^{(9)}$	0.7413$^{(10)}$	0.6455$^{(6)}$	0.1541$^{(1)}$	0.6459$^{(7)}$	0.6018$^{(3)}$	0.6802$^{(8)}$	0.7819$^{(11)}$	1.3658$^{(12)}$	0.6061$^{(4)}$
		θ	0.0858$^{(2)}$	0.0905$^{(3)}$	0.1013$^{(8)}$	0.1019$^{(10)}$	0.0941$^{(5)}$	0.0661$^{(1)}$	0.0929$^{(4)}$	0.1020$^{(11)}$	0.1015$^{(9)}$	0.1007$^{(7)}$	0.1373$^{(12)}$	0.0967$^{(6)}$
	Sum Rank		12$^{(2)}$	24$^{(3)}$	51$^{(8.5)}$	60$^{(11)}$	33$^{(5.5)}$	6$^{(1)}$	33$^{(5.5)}$	42$^{(7)}$	51$^{(8.5)}$	54$^{(10)}$	72$^{(12)}$	30$^{(4)}$
100	MSE	α	0.0449$^{(2)}$	0.0485$^{(4)}$	0.0676$^{(9)}$	0.0704$^{(10)}$	0.0534$^{(7)}$	0.0035$^{(1)}$	0.0522$^{(6)}$	0.0485$^{(5)}$	0.0590$^{(8)}$	0.0799$^{(11)}$	0.2397$^{(12)}$	0.0480$^{(3)}$
		θ	0.0048$^{(2)}$	0.0054$^{(3)}$	0.0068$^{(8)}$	0.0068$^{(10)}$	0.0058$^{(5)}$	0.0030$^{(1)}$	0.0056$^{(4)}$	0.0070$^{(11)}$	0.0068$^{(9)}$	0.0067$^{(7)}$	0.0128$^{(12)}$	0.0062$^{(6)}$
	ABB	α	0.2118$^{(2)}$	0.2203$^{(4)}$	0.2599$^{(9)}$	0.2653$^{(10)}$	0.2311$^{(7)}$	0.0590$^{(1)}$	0.2285$^{(6)}$	0.2203$^{(5)}$	0.2429$^{(8)}$	0.2827$^{(11)}$	0.4896$^{(12)}$	0.2191$^{(3)}$
		θ	0.0696$^{(2)}$	0.0732$^{(3)}$	0.0822$^{(8)}$	0.0825$^{(10)}$	0.0760$^{(5)}$	0.0545$^{(1)}$	0.0748$^{(4)}$	0.0836$^{(11)}$	0.0822$^{(9)}$	0.0817$^{(7)}$	0.1133$^{(12)}$	0.0788$^{(6)}$
	MRE	α	0.5296$^{(2)}$	0.5507$^{(4)}$	0.6498$^{(9)}$	0.6633$^{(10)}$	0.5777$^{(7)}$	0.1475$^{(1)}$	0.5713$^{(6)}$	0.5507$^{(5)}$	0.6072$^{(8)}$	0.7067$^{(11)}$	1.2240$^{(12)}$	0.5479$^{(3)}$
		θ	0.0773$^{(2)}$	0.0813$^{(3)}$	0.0913$^{(8)}$	0.0917$^{(10)}$	0.0844$^{(5)}$	0.0605$^{(1)}$	0.0832$^{(4)}$	0.0929$^{(11)}$	0.0913$^{(9)}$	0.0908$^{(7)}$	0.1259$^{(12)}$	0.0875$^{(6)}$
	Sum Rank		12$^{(2)}$	21$^{(3)}$	51$^{(8.5)}$	60$^{(11)}$	36$^{(6)}$	6$^{(1)}$	30$^{(5)}$	48$^{(7)}$	51$^{(8.5)}$	54$^{(10)}$	72$^{(12)}$	27$^{(4)}$

n		Parameter	MLE	ADE	CvME	LSE	WLSE	KME	MPSE	MSADE	MSALDE	ADLTE	ADLTSOE	ADRTE
250	MSE	α	0.0315$^{(2)}$	0.0351$^{(4)}$	0.0484$^{(9)}$	0.0497$^{(10)}$	0.0379$^{(7)}$	0.0029$^{(1)}$	0.0359$^{(6)}$	0.0354$^{(5)}$	0.0419$^{(8)}$	0.0574$^{(11)}$	0.1761$^{(12)}$	0.0346$^{(3)}$
		θ	0.0034$^{(2)}$	0.0038$^{(3)}$	0.0048$^{(8)}$	0.0048$^{(9)}$	0.0040$^{(5)}$	0.0022$^{(1)}$	0.0038$^{(4)}$	0.0051$^{(11)}$	0.0048$^{(10)}$	0.0047$^{(7)}$	0.0097$^{(12)}$	0.0044$^{(6)}$
	ABB	α	0.1774$^{(2)}$	0.1872$^{(4)}$	0.2200$^{(9)}$	0.2229$^{(10)}$	0.1946$^{(7)}$	0.0543$^{(1)}$	0.1895$^{(6)}$	0.1881$^{(5)}$	0.2047$^{(8)}$	0.2397$^{(11)}$	0.4197$^{(12)}$	0.1861$^{(3)}$
		θ	0.0584$^{(2)}$	0.0616$^{(3)}$	0.0695$^{(8)}$	0.0696$^{(9)}$	0.0632$^{(5)}$	0.0465$^{(1)}$	0.0620$^{(4)}$	0.0712$^{(11)}$	0.0696$^{(10)}$	0.0688$^{(7)}$	0.0986$^{(12)}$	0.0666$^{(6)}$
	MRE	α	0.4435$^{(2)}$	0.4681$^{(4)}$	0.5499$^{(9)}$	0.5572$^{(10)}$	0.4864$^{(7)}$	0.1357$^{(1)}$	0.4737$^{(6)}$	0.4702$^{(5)}$	0.5118$^{(8)}$	0.5991$^{(11)}$	1.0491$^{(12)}$	0.4653$^{(3)}$
		θ	0.0649$^{(2)}$	0.0685$^{(3)}$	0.0772$^{(8)}$	0.0773$^{(9)}$	0.0702$^{(5)}$	0.0516$^{(1)}$	0.0689$^{(4)}$	0.0792$^{(11)}$	0.0773$^{(10)}$	0.0765$^{(7)}$	0.1096$^{(12)}$	0.0740$^{(6)}$
	Sum Rank		12$^{(2)}$	21$^{(3)}$	51$^{(8)}$	57$^{(11)}$	36$^{(6)}$	6$^{(1)}$	30$^{(5)}$	48$^{(7)}$	54$^{(9.5)}$	54$^{(9.5)}$	72$^{(12)}$	27$^{(4)}$
500	MSE	α	0.0218$^{(2)}$	0.0248$^{(5)}$	0.0339$^{(9)}$	0.0348$^{(10)}$	0.0263$^{(7)}$	0.0025$^{(1)}$	0.0240$^{(3)}$	0.0252$^{(6)}$	0.0289$^{(8)}$	0.0406$^{(11)}$	0.1276$^{(12)}$	0.0246$^{(4)}$
		θ	0.0024$^{(2)}$	0.0027$^{(4)}$	0.0034$^{(9)}$	0.0034$^{(10)}$	0.0028$^{(5)}$	0.0016$^{(1)}$	0.0026$^{(3)}$	0.0036$^{(11)}$	0.0033$^{(8)}$	0.0033$^{(7)}$	0.0071$^{(12)}$	0.0031$^{(6)}$
	ABB	α	0.1477$^{(2)}$	0.1574$^{(5)}$	0.1842$^{(9)}$	0.1865$^{(10)}$	0.1621$^{(7)}$	0.0497$^{(1)}$	0.1548$^{(3)}$	0.1586$^{(6)}$	0.1700$^{(8)}$	0.2014$^{(11)}$	0.3573$^{(12)}$	0.1567$^{(4)}$
		θ	0.0488$^{(2)}$	0.0518$^{(4)}$	0.0582$^{(9)}$	0.0582$^{(10)}$	0.0526$^{(5)}$	0.0396$^{(1)}$	0.0512$^{(3)}$	0.0598$^{(11)}$	0.0578$^{(8)}$	0.0576$^{(7)}$	0.0841$^{(12)}$	0.0555$^{(6)}$
	MRE	α	0.3693$^{(2)}$	0.3936$^{(5)}$	0.4605$^{(9)}$	0.4661$^{(10)}$	0.4052$^{(7)}$	0.1243$^{(1)}$	0.3869$^{(3)}$	0.3965$^{(6)}$	0.4249$^{(8)}$	0.5035$^{(11)}$	0.8932$^{(12)}$	0.3918$^{(4)}$
		θ	0.0542$^{(2)}$	0.0575$^{(4)}$	0.0646$^{(9)}$	0.0647$^{(10)}$	0.0585$^{(5)}$	0.0440$^{(1)}$	0.0569$^{(3)}$	0.0665$^{(11)}$	0.0643$^{(8)}$	0.0641$^{(7)}$	0.0934$^{(12)}$	0.0616$^{(6)}$
	Sum Rank		12$^{(2)}$	27$^{(4)}$	54$^{(9.5)}$	60$^{(11)}$	36$^{(6)}$	6$^{(1)}$	18$^{(3)}$	51$^{(8)}$	48$^{(7)}$	54$^{(9.5)}$	72$^{(12)}$	30$^{(5)}$

Table 4. The MSE, ABB and MRE values for ψ_{S2}.

n		Parameter	MLE	ADE	CvME	LSE	WLSE	KME	MPSE	MSADE	MSALDE	ADLTE	ADLTSOE	ADRTE
25	MSE	α	0.1657$^{(3)}$	0.1780$^{(4)}$	0.2017$^{(5)}$	0.2497$^{(11)}$	0.2098$^{(7)}$	0.0230$^{(1)}$	0.2495$^{(10)}$	0.1424$^{(2)}$	0.2272$^{(9)}$	0.2150$^{(8)}$	0.3688$^{(12)}$	0.2092$^{(6)}$
		θ	0.0664$^{(3)}$	0.0729$^{(4)}$	0.0845$^{(6)}$	0.0959$^{(10)}$	0.0890$^{(9)}$	0.0215$^{(1)}$	0.0868$^{(8)}$	0.0591$^{(2)}$	0.0862$^{(7)}$	0.0808$^{(5)}$	0.1162$^{(12)}$	0.1010$^{(11)}$
	ABB	α	0.4070$^{(3)}$	0.4219$^{(4)}$	0.4491$^{(5)}$	0.4997$^{(11)}$	0.4581$^{(7)}$	0.1517$^{(1)}$	0.4995$^{(10)}$	0.3774$^{(2)}$	0.4767$^{(9)}$	0.4637$^{(8)}$	0.6073$^{(12)}$	0.4574$^{(6)}$
		θ	0.2578$^{(3)}$	0.2700$^{(4)}$	0.2907$^{(6)}$	0.3096$^{(10)}$	0.2984$^{(9)}$	0.1468$^{(1)}$	0.2946$^{(8)}$	0.2431$^{(2)}$	0.2936$^{(7)}$	0.2842$^{(5)}$	0.3409$^{(12)}$	0.3177$^{(11)}$
	MRE	α	0.5815$^{(3)}$	0.6027$^{(4)}$	0.6416$^{(5)}$	0.7138$^{(11)}$	0.6544$^{(7)}$	0.2168$^{(1)}$	0.7136$^{(10)}$	0.5392$^{(2)}$	0.6810$^{(9)}$	0.6625$^{(8)}$	0.8676$^{(12)}$	0.6534$^{(6)}$
		θ	0.1611$^{(3)}$	0.1687$^{(4)}$	0.1817$^{(6)}$	0.1935$^{(10)}$	0.1865$^{(9)}$	0.0917$^{(1)}$	0.1841$^{(8)}$	0.1519$^{(2)}$	0.1835$^{(7)}$	0.1776$^{(5)}$	0.2130$^{(12)}$	0.1986$^{(11)}$
	Sum Rank		18$^{(3)}$	24$^{(4)}$	33$^{(5)}$	63$^{(11)}$	48$^{(7.5)}$	6$^{(1)}$	54$^{(10)}$	12$^{(2)}$	48$^{(7.5)}$	39$^{(6)}$	72$^{(12)}$	51$^{(9)}$
50	MSE	α	0.1190$^{(3)}$	0.1289$^{(4)}$	0.1497$^{(6)}$	0.1698$^{(10)}$	0.1416$^{(5)}$	0.0179$^{(1)}$	0.1612$^{(9)}$	0.1129$^{(2)}$	0.1698$^{(11)}$	0.1551$^{(8)}$	0.2892$^{(12)}$	0.1539$^{(7)}$
		θ	0.0446$^{(2)}$	0.0498$^{(4)}$	0.0597$^{(8)}$	0.0654$^{(10)}$	0.0562$^{(6)}$	0.0158$^{(1)}$	0.0572$^{(7)}$	0.0469$^{(3)}$	0.0621$^{(9)}$	0.0539$^{(5)}$	0.0860$^{(12)}$	0.0695$^{(11)}$
	ABB	α	0.3450$^{(3)}$	0.3590$^{(4)}$	0.3869$^{(6)}$	0.4121$^{(10)}$	0.3763$^{(5)}$	0.1340$^{(1)}$	0.4015$^{(9)}$	0.3361$^{(2)}$	0.4121$^{(11)}$	0.3938$^{(8)}$	0.5378$^{(12)}$	0.3923$^{(7)}$
		θ	0.2112$^{(2)}$	0.2232$^{(4)}$	0.2443$^{(8)}$	0.2557$^{(10)}$	0.2371$^{(6)}$	0.1257$^{(1)}$	0.2392$^{(7)}$	0.2166$^{(3)}$	0.2491$^{(9)}$	0.2321$^{(5)}$	0.2932$^{(12)}$	0.2637$^{(11)}$
	MRE	α	0.4928$^{(3)}$	0.5129$^{(4)}$	0.5527$^{(6)}$	0.5887$^{(10)}$	0.5375$^{(5)}$	0.1914$^{(1)}$	0.5735$^{(9)}$	0.4801$^{(2)}$	0.5887$^{(11)}$	0.5626$^{(8)}$	0.7682$^{(12)}$	0.5604$^{(7)}$
		θ	0.1320$^{(2)}$	0.1395$^{(4)}$	0.1527$^{(8)}$	0.1598$^{(10)}$	0.1482$^{(6)}$	0.0785$^{(1)}$	0.1495$^{(7)}$	0.1354$^{(3)}$	0.1557$^{(9)}$	0.1450$^{(5)}$	0.1833$^{(12)}$	0.1648$^{(11)}$
	Sum Rank		15$^{(2.5)}$	24$^{(4)}$	42$^{(7)}$	60$^{(10.5)}$	33$^{(5)}$	6$^{(1)}$	48$^{(8)}$	15$^{(2.5)}$	60$^{(10.5)}$	39$^{(6)}$	72$^{(12)}$	54$^{(9)}$
75	MSE	α	0.0927$^{(2)}$	0.1023$^{(4)}$	0.1194$^{(6)}$	0.1300$^{(10)}$	0.1088$^{(5)}$	0.0146$^{(1)}$	0.1224$^{(8)}$	0.0947$^{(3)}$	0.1373$^{(11)}$	0.1203$^{(7)}$	0.2394$^{(12)}$	0.1237$^{(9)}$
		θ	0.0341$^{(2)}$	0.0377$^{(3)}$	0.0458$^{(8)}$	0.0485$^{(10)}$	0.0411$^{(6)}$	0.0126$^{(1)}$	0.0423$^{(7)}$	0.0386$^{(4)}$	0.0485$^{(9)}$	0.0406$^{(5)}$	0.0682$^{(12)}$	0.0529$^{(11)}$
	ABB	α	0.3045$^{(2)}$	0.3199$^{(4)}$	0.3455$^{(6)}$	0.3605$^{(10)}$	0.3298$^{(5)}$	0.1210$^{(1)}$	0.3498$^{(8)}$	0.3077$^{(3)}$	0.3705$^{(11)}$	0.3469$^{(7)}$	0.4892$^{(12)}$	0.3518$^{(9)}$
		θ	0.1846$^{(2)}$	0.1941$^{(3)}$	0.2141$^{(8)}$	0.2203$^{(10)}$	0.2027$^{(6)}$	0.1121$^{(1)}$	0.2057$^{(7)}$	0.1965$^{(4)}$	0.2202$^{(9)}$	0.2014$^{(5)}$	0.2611$^{(12)}$	0.2301$^{(11)}$
	MRE	α	0.4350$^{(2)}$	0.4570$^{(4)}$	0.4936$^{(6)}$	0.5150$^{(10)}$	0.4712$^{(5)}$	0.1728$^{(1)}$	0.4997$^{(8)}$	0.4396$^{(3)}$	0.5293$^{(11)}$	0.4955$^{(7)}$	0.6989$^{(12)}$	0.5025$^{(9)}$
		θ	0.1154$^{(2)}$	0.1213$^{(3)}$	0.1338$^{(8)}$	0.1377$^{(10)}$	0.1267$^{(6)}$	0.0700$^{(1)}$	0.1285$^{(7)}$	0.1228$^{(4)}$	0.1376$^{(9)}$	0.1259$^{(5)}$	0.1632$^{(12)}$	0.1438$^{(11)}$
	Sum Rank		12$^{(2)}$	21$^{(3.5)}$	42$^{(7)}$	60$^{(10)}$	33$^{(5)}$	6$^{(1)}$	45$^{(8)}$	21$^{(3.5)}$	60$^{(10)}$	36$^{(6)}$	72$^{(12)}$	60$^{(10)}$
100	MSE	α	0.0781$^{(2)}$	0.0855$^{(4)}$	0.0994$^{(7)}$	0.1071$^{(10)}$	0.0896$^{(5)}$	0.0125$^{(1)}$	0.0990$^{(6)}$	0.0827$^{(3)}$	0.1152$^{(11)}$	0.1002$^{(8)}$	0.2053$^{(12)}$	0.1031$^{(9)}$
		θ	0.0277$^{(2)}$	0.0306$^{(3)}$	0.0373$^{(8)}$	0.0391$^{(9)}$	0.0329$^{(4)}$	0.0107$^{(1)}$	0.0338$^{(7)}$	0.0333$^{(6)}$	0.0401$^{(10)}$	0.0332$^{(5)}$	0.0566$^{(12)}$	0.0429$^{(11)}$
	ABB	α	0.2794$^{(2)}$	0.2924$^{(4)}$	0.3153$^{(7)}$	0.3272$^{(10)}$	0.2993$^{(5)}$	0.1120$^{(1)}$	0.3146$^{(6)}$	0.2876$^{(3)}$	0.3395$^{(11)}$	0.3165$^{(8)}$	0.4531$^{(12)}$	0.3211$^{(9)}$
		θ	0.1666$^{(2)}$	0.1750$^{(3)}$	0.1931$^{(8)}$	0.1978$^{(9)}$	0.1815$^{(4)}$	0.1033$^{(1)}$	0.1839$^{(7)}$	0.1824$^{(6)}$	0.2002$^{(10)}$	0.1822$^{(5)}$	0.2379$^{(12)}$	0.2072$^{(11)}$
	MRE	α	0.3992$^{(2)}$	0.4177$^{(4)}$	0.4504$^{(7)}$	0.4674$^{(10)}$	0.4276$^{(5)}$	0.1600$^{(1)}$	0.4495$^{(6)}$	0.4108$^{(3)}$	0.4850$^{(11)}$	0.4522$^{(8)}$	0.6473$^{(12)}$	0.4587$^{(9)}$
		θ	0.1041$^{(2)}$	0.1094$^{(3)}$	0.1207$^{(8)}$	0.1236$^{(9)}$	0.1134$^{(4)}$	0.0646$^{(1)}$	0.1150$^{(7)}$	0.1140$^{(6)}$	0.1251$^{(10)}$	0.1139$^{(5)}$	0.1487$^{(12)}$	0.1295$^{(11)}$
	Sum Rank		12$^{(2)}$	21$^{(3)}$	45$^{(8)}$	57$^{(9)}$	27$^{(4.5)}$	6$^{(1)}$	39$^{(6.5)}$	27$^{(4.5)}$	63$^{(11)}$	39$^{(6.5)}$	72$^{(12)}$	60$^{(10)}$
250	MSE	α	0.0573$^{(2)}$	0.0621$^{(3)}$	0.0727$^{(7)}$	0.0772$^{(10)}$	0.0651$^{(5)}$	0.0098$^{(1)}$	0.0708$^{(6)}$	0.0632$^{(4)}$	0.0853$^{(11)}$	0.0738$^{(8)}$	0.1630$^{(12)}$	0.0759$^{(9)}$
		θ	0.0198$^{(2)}$	0.0217$^{(3)}$	0.0266$^{(8)}$	0.0279$^{(9)}$	0.0234$^{(4)}$	0.0083$^{(1)}$	0.0236$^{(5)}$	0.0250$^{(7)}$	0.0288$^{(10)}$	0.0236$^{(6)}$	0.0428$^{(12)}$	0.0309$^{(11)}$
	ABB	α	0.2393$^{(2)}$	0.2492$^{(3)}$	0.2696$^{(7)}$	0.2778$^{(10)}$	0.2551$^{(5)}$	0.0990$^{(1)}$	0.2661$^{(6)}$	0.2513$^{(4)}$	0.2920$^{(11)}$	0.2717$^{(8)}$	0.4037$^{(12)}$	0.2755$^{(9)}$
		θ	0.1408$^{(2)}$	0.1473$^{(3)}$	0.1630$^{(8)}$	0.1669$^{(9)}$	0.1531$^{(4)}$	0.0911$^{(1)}$	0.1537$^{(5)}$	0.1582$^{(7)}$	0.1696$^{(10)}$	0.1538$^{(6)}$	0.2068$^{(12)}$	0.1757$^{(11)}$
	MRE	α	0.3419$^{(2)}$	0.3560$^{(3)}$	0.3852$^{(7)}$	0.3969$^{(10)}$	0.3645$^{(5)}$	0.1414$^{(1)}$	0.3801$^{(6)}$	0.3591$^{(4)}$	0.4171$^{(11)}$	0.3881$^{(8)}$	0.5767$^{(12)}$	0.3936$^{(9)}$
		θ	0.0880$^{(2)}$	0.0920$^{(3)}$	0.1019$^{(8)}$	0.1043$^{(9)}$	0.0957$^{(4)}$	0.0569$^{(1)}$	0.0960$^{(5)}$	0.0989$^{(7)}$	0.1060$^{(10)}$	0.0961$^{(6)}$	0.1293$^{(12)}$	0.1098$^{(11)}$
	Sum Rank		12$^{(2)}$	18$^{(3)}$	45$^{(8)}$	57$^{(9)}$	27$^{(4)}$	6$^{(1)}$	33$^{(5.5)}$	33$^{(5.5)}$	63$^{(11)}$	42$^{(7)}$	72$^{(12)}$	60$^{(10)}$
500	MSE	α	0.0410$^{(2)}$	0.0444$^{(3)}$	0.0522$^{(7)}$	0.0549$^{(9)}$	0.0459$^{(4)}$	0.0077$^{(1)}$	0.0485$^{(6)}$	0.0464$^{(5)}$	0.0610$^{(11)}$	0.0526$^{(8)}$	0.1253$^{(12)}$	0.0555$^{(10)}$
		θ	0.0138$^{(2)}$	0.0152$^{(3)}$	0.0187$^{(8)}$	0.0194$^{(9)}$	0.0160$^{(4)}$	0.0063$^{(1)}$	0.0161$^{(5)}$	0.0182$^{(7)}$	0.0201$^{(10)}$	0.0165$^{(6)}$	0.0311$^{(12)}$	0.0215$^{(11)}$
	ABB	α	0.2026$^{(2)}$	0.2108$^{(3)}$	0.2286$^{(7)}$	0.2343$^{(9)}$	0.2143$^{(4)}$	0.0880$^{(1)}$	0.2203$^{(6)}$	0.2155$^{(5)}$	0.2469$^{(11)}$	0.2294$^{(8)}$	0.3540$^{(12)}$	0.2355$^{(10)}$
		θ	0.1176$^{(2)}$	0.1233$^{(3)}$	0.1367$^{(8)}$	0.1392$^{(9)}$	0.1264$^{(4)}$	0.0796$^{(1)}$	0.1270$^{(5)}$	0.1350$^{(7)}$	0.1417$^{(10)}$	0.1285$^{(6)}$	0.1762$^{(12)}$	0.1468$^{(11)}$
	MRE	α	0.2894$^{(2)}$	0.3011$^{(3)}$	0.3265$^{(7)}$	0.3347$^{(9)}$	0.3061$^{(4)}$	0.1258$^{(1)}$	0.3147$^{(6)}$	0.3078$^{(5)}$	0.3528$^{(11)}$	0.3278$^{(8)}$	0.5058$^{(12)}$	0.3364$^{(10)}$
		θ	0.0735$^{(2)}$	0.0771$^{(3)}$	0.0854$^{(8)}$	0.0870$^{(9)}$	0.0790$^{(4)}$	0.0497$^{(1)}$	0.0794$^{(5)}$	0.0844$^{(7)}$	0.0886$^{(10)}$	0.0803$^{(6)}$	0.1102$^{(12)}$	0.0917$^{(11)}$
	Sum Rank		12$^{(2)}$	18$^{(3)}$	45$^{(8)}$	54$^{(9)}$	24$^{(4)}$	6$^{(1)}$	33$^{(5)}$	36$^{(6)}$	63$^{(10.5)}$	42$^{(7)}$	72$^{(12)}$	63$^{(10.5)}$

Table 5. The MSE, ABB and MRE values for ψ_{S3}.

n		Parameter	MLE	ADE	CvME	LSE	WLSE	KME	MPSE	MSADE	MSALDE	ADLTE	ADLTSOE	ADRTE
25	MSE	α	4.3188$^{(6)}$	5.1364$^{(9)}$	4.2338$^{(4)}$	4.3410$^{(7)}$	5.9649$^{(12)}$	0.0005$^{(1)}$	4.2869$^{(5)}$	0.0018$^{(2)}$	3.0856$^{(3)}$	4.9414$^{(8)}$	5.7228$^{(11)}$	5.3794$^{(10)}$
		θ	0.1070$^{(9)}$	0.0857$^{(8)}$	0.0810$^{(6)}$	0.0595$^{(3)}$	0.1187$^{(11)}$	0.0229$^{(1)}$	0.0673$^{(4)}$	0.0361$^{(2)}$	0.0692$^{(5)}$	0.0846$^{(7)}$	0.1499$^{(12)}$	0.1117$^{(10)}$
	ABB	α	2.0782$^{(6)}$	2.2664$^{(9)}$	2.0576$^{(4)}$	2.0835$^{(7)}$	2.4423$^{(12)}$	0.0219$^{(1)}$	2.0705$^{(5)}$	0.0429$^{(2)}$	1.7566$^{(3)}$	2.2229$^{(8)}$	2.3922$^{(11)}$	2.3194$^{(10)}$
		θ	0.3270$^{(9)}$	0.2928$^{(8)}$	0.2846$^{(6)}$	0.2439$^{(3)}$	0.3445$^{(11)}$	0.1514$^{(1)}$	0.2594$^{(4)}$	0.1901$^{(2)}$	0.2632$^{(5)}$	0.2909$^{(7)}$	0.3871$^{(12)}$	0.3343$^{(10)}$
	MRE	α	0.7697$^{(6)}$	0.8394$^{(9)}$	0.7621$^{(4)}$	0.7717$^{(7)}$	0.9046$^{(12)}$	0.0081$^{(1)}$	0.7668$^{(5)}$	0.0159$^{(2)}$	0.6506$^{(3)}$	0.8233$^{(8)}$	0.8860$^{(11)}$	0.8590$^{(10)}$
		θ	0.2336$^{(9)}$	0.2091$^{(8)}$	0.2033$^{(6)}$	0.1742$^{(3)}$	0.2461$^{(11)}$	0.1081$^{(1)}$	0.1853$^{(4)}$	0.1358$^{(2)}$	0.1880$^{(5)}$	0.2078$^{(7)}$	0.2765$^{(12)}$	0.2388$^{(10)}$
	Sum Rank		45$^{(7.5)}$	51$^{(9)}$	30$^{(5.5)}$	30$^{(5.5)}$	69$^{(11.5)}$	6$^{(1)}$	27$^{(4)}$	12$^{(2)}$	24$^{(3)}$	45$^{(7.5)}$	69$^{(11.5)}$	60$^{(10)}$
50	MSE	α	3.7889$^{(5)}$	4.6039$^{(9)}$	4.0020$^{(6)}$	4.2418$^{(7)}$	5.1397$^{(11)}$	0.0004$^{(1)}$	3.5630$^{(4)}$	0.0017$^{(2)}$	2.9871$^{(3)}$	4.3564$^{(8)}$	5.1951$^{(12)}$	4.9673$^{(10)}$
		θ	0.0932$^{(10)}$	0.0795$^{(8)}$	0.0707$^{(6)}$	0.0559$^{(3)}$	0.1072$^{(11)}$	0.0163$^{(1)}$	0.0563$^{(4)}$	0.0280$^{(2)}$	0.0608$^{(5)}$	0.0744$^{(7)}$	0.1341$^{(12)}$	0.0825$^{(9)}$
	ABB	α	1.9465$^{(5)}$	2.1457$^{(9)}$	2.0005$^{(6)}$	2.0596$^{(7)}$	2.2671$^{(11)}$	0.0188$^{(1)}$	1.8876$^{(4)}$	0.0407$^{(2)}$	1.7283$^{(3)}$	2.0872$^{(8)}$	2.2793$^{(12)}$	2.2287$^{(10)}$
		θ	0.3053$^{(10)}$	0.2819$^{(8)}$	0.2660$^{(6)}$	0.2364$^{(3)}$	0.3274$^{(11)}$	0.1276$^{(1)}$	0.2374$^{(4)}$	0.1674$^{(2)}$	0.2466$^{(5)}$	0.2727$^{(7)}$	0.3662$^{(12)}$	0.2872$^{(9)}$
	MRE	α	0.7209$^{(5)}$	0.7947$^{(9)}$	0.7409$^{(6)}$	0.7628$^{(7)}$	0.8397$^{(11)}$	0.0069$^{(1)}$	0.6991$^{(4)}$	0.0151$^{(2)}$	0.6401$^{(3)}$	0.7730$^{(8)}$	0.8442$^{(12)}$	0.8255$^{(10)}$
		θ	0.2181$^{(10)}$	0.2014$^{(8)}$	0.1900$^{(6)}$	0.1688$^{(3)}$	0.2338$^{(11)}$	0.0911$^{(1)}$	0.1695$^{(4)}$	0.1196$^{(2)}$	0.1761$^{(5)}$	0.1948$^{(7)}$	0.2616$^{(12)}$	0.2051$^{(9)}$
	Sum Rank		45$^{(7.5)}$	51$^{(9)}$	36$^{(6)}$	30$^{(5)}$	66$^{(11)}$	6$^{(1)}$	24$^{(3.5)}$	12$^{(2)}$	24$^{(3.5)}$	45$^{(7.5)}$	72$^{(12)}$	57$^{(10)}$
75	MSE	α	3.3815$^{(5)}$	4.1548$^{(9)}$	3.7945$^{(6)}$	4.0849$^{(8)}$	4.5537$^{(10)}$	0.0003$^{(1)}$	3.0567$^{(4)}$	0.0016$^{(2)}$	2.8730$^{(3)}$	3.9440$^{(7)}$	4.8331$^{(12)}$	4.5855$^{(11)}$
		θ	0.0774$^{(10)}$	0.0723$^{(8)}$	0.0633$^{(6)}$	0.0545$^{(5)}$	0.0974$^{(11)}$	0.0124$^{(1)}$	0.0477$^{(3)}$	0.0231$^{(2)}$	0.0543$^{(4)}$	0.0679$^{(7)}$	0.1170$^{(12)}$	0.0740$^{(9)}$
	ABB	α	1.8389$^{(5)}$	2.0383$^{(9)}$	1.9479$^{(6)}$	2.0211$^{(8)}$	2.1339$^{(10)}$	0.0167$^{(1)}$	1.7483$^{(4)}$	0.0398$^{(2)}$	1.6950$^{(3)}$	1.9859$^{(7)}$	2.1984$^{(12)}$	2.1414$^{(11)}$
		θ	0.2782$^{(10)}$	0.2688$^{(8)}$	0.2516$^{(6)}$	0.2335$^{(5)}$	0.3120$^{(11)}$	0.1115$^{(1)}$	0.2183$^{(3)}$	0.1521$^{(2)}$	0.2330$^{(4)}$	0.2606$^{(7)}$	0.3420$^{(12)}$	0.2721$^{(9)}$
	MRE	α	0.6811$^{(5)}$	0.7549$^{(9)}$	0.7215$^{(6)}$	0.7486$^{(8)}$	0.7904$^{(10)}$	0.0062$^{(1)}$	0.6475$^{(4)}$	0.0147$^{(2)}$	0.6278$^{(3)}$	0.7355$^{(7)}$	0.8142$^{(12)}$	0.7931$^{(11)}$
		θ	0.1987$^{(10)}$	0.1920$^{(8)}$	0.1797$^{(6)}$	0.1668$^{(5)}$	0.2229$^{(11)}$	0.0797$^{(1)}$	0.1559$^{(3)}$	0.1087$^{(2)}$	0.1664$^{(4)}$	0.1862$^{(7)}$	0.2443$^{(12)}$	0.1943$^{(9)}$
	Sum Rank		45$^{(8)}$	51$^{(9)}$	36$^{(5)}$	39$^{(6)}$	63$^{(11)}$	6$^{(1)}$	21$^{(3.5)}$	12$^{(2)}$	21$^{(3.5)}$	42$^{(7)}$	72$^{(12)}$	60$^{(10)}$
100	MSE	α	3.0566$^{(5)}$	3.7699$^{(9)}$	3.5218$^{(6)}$	3.7580$^{(8)}$	4.0905$^{(10)}$	0.0002$^{(1)}$	2.6803$^{(3)}$	0.0015$^{(2)}$	2.7459$^{(4)}$	3.6065$^{(7)}$	4.5154$^{(12)}$	4.2846$^{(11)}$
		θ	0.0673$^{(9)}$	0.0645$^{(8)}$	0.0553$^{(6)}$	0.0491$^{(4)}$	0.0913$^{(11)}$	0.0102$^{(1)}$	0.0420$^{(3)}$	0.0197$^{(2)}$	0.0499$^{(5)}$	0.0644$^{(7)}$	0.1041$^{(12)}$	0.0676$^{(10)}$
	ABB	α	1.7483$^{(5)}$	1.9416$^{(9)}$	1.8766$^{(6)}$	1.9386$^{(8)}$	2.0225$^{(10)}$	0.0154$^{(1)}$	1.6372$^{(3)}$	0.0389$^{(2)}$	1.6571$^{(4)}$	1.8991$^{(7)}$	2.1249$^{(12)}$	2.0699$^{(11)}$
		θ	0.2595$^{(9)}$	0.2540$^{(8)}$	0.2351$^{(6)}$	0.2216$^{(4)}$	0.3022$^{(11)}$	0.1008$^{(1)}$	0.2049$^{(3)}$	0.1404$^{(2)}$	0.2233$^{(5)}$	0.2538$^{(7)}$	0.3226$^{(12)}$	0.2600$^{(10)}$
	MRE	α	0.6475$^{(5)}$	0.7191$^{(9)}$	0.6950$^{(6)}$	0.7180$^{(8)}$	0.7491$^{(10)}$	0.0057$^{(1)}$	0.6064$^{(3)}$	0.0144$^{(2)}$	0.6137$^{(4)}$	0.7034$^{(7)}$	0.7870$^{(12)}$	0.7666$^{(11)}$
		θ	0.1854$^{(9)}$	0.1814$^{(8)}$	0.1679$^{(6)}$	0.1583$^{(4)}$	0.2158$^{(11)}$	0.0720$^{(1)}$	0.1464$^{(3)}$	0.1003$^{(2)}$	0.1595$^{(5)}$	0.1813$^{(7)}$	0.2304$^{(12)}$	0.1857$^{(10)}$
	Sum Rank		42$^{(7.5)}$	51$^{(9)}$	36$^{(5.5)}$	36$^{(5.5)}$	63$^{(10.5)}$	6$^{(1)}$	18$^{(3)}$	12$^{(2)}$	27$^{(4)}$	42$^{(7.5)}$	72$^{(12)}$	63$^{(10.5)}$

n		Parameter	MLE	ADE	CvME	LSE	WLSE	KME	MPSE	MSADE	MSALDE	ADLTE	ADLTSOE	ADRTE
250	MSE	α	2.6097$^{(5)}$	3.2658$^{(8)}$	3.1646$^{(7)}$	3.3793$^{(9)}$	3.5041$^{(10)}$	0.0002$^{(1)}$	2.2044$^{(3)}$	0.0017$^{(2)}$	2.4222$^{(4)}$	3.0965$^{(6)}$	4.0817$^{(12)}$	3.9196$^{(11)}$
		θ	0.0541$^{(8)}$	0.0552$^{(9)}$	0.0481$^{(6)}$	0.0442$^{(5)}$	0.0750$^{(11)}$	0.0073$^{(1)}$	0.0331$^{(3)}$	0.0155$^{(2)}$	0.0428$^{(4)}$	0.0535$^{(7)}$	0.0878$^{(12)}$	0.0600$^{(10)}$
	ABB	α	1.6154$^{(5)}$	1.8071$^{(8)}$	1.7789$^{(7)}$	1.8383$^{(9)}$	1.8719$^{(10)}$	0.0137$^{(1)}$	1.4847$^{(3)}$	0.0410$^{(2)}$	1.5564$^{(4)}$	1.7597$^{(6)}$	2.0203$^{(12)}$	1.9798$^{(11)}$
		θ	0.2325$^{(8)}$	0.2350$^{(9)}$	0.2194$^{(6)}$	0.2103$^{(5)}$	0.2739$^{(11)}$	0.0854$^{(1)}$	0.1819$^{(3)}$	0.1244$^{(2)}$	0.2068$^{(4)}$	0.2313$^{(7)}$	0.2964$^{(12)}$	0.2449$^{(10)}$
	MRE	α	0.5983$^{(5)}$	0.6693$^{(8)}$	0.6589$^{(7)}$	0.6809$^{(9)}$	0.6933$^{(10)}$	0.0051$^{(1)}$	0.5499$^{(3)}$	0.0152$^{(2)}$	0.5764$^{(4)}$	0.6517$^{(6)}$	0.7483$^{(12)}$	0.7333$^{(11)}$
		θ	0.1661$^{(8)}$	0.1678$^{(9)}$	0.1567$^{(6)}$	0.1502$^{(5)}$	0.1956$^{(11)}$	0.0610$^{(1)}$	0.1299$^{(3)}$	0.0888$^{(2)}$	0.1477$^{(4)}$	0.1652$^{(7)}$	0.2117$^{(12)}$	0.1749$^{(10)}$
	Sum Rank		39$^{(6)}$	51$^{(9)}$	39$^{(6)}$	42$^{(8)}$	63$^{(10.5)}$	6$^{(1)}$	18$^{(3)}$	12$^{(2)}$	24$^{(4)}$	39$^{(6)}$	72$^{(12)}$	63$^{(10.5)}$
500	MSE	α	2.1208$^{(5)}$	2.6935$^{(7)}$	2.7327$^{(8)}$	2.9008$^{(10)}$	2.8598$^{(9)}$	0.0002$^{(1)}$	1.7418$^{(3)}$	0.0021$^{(2)}$	2.0295$^{(4)}$	2.5327$^{(6)}$	3.5967$^{(12)}$	3.4544$^{(11)}$
		θ	0.0407$^{(7)}$	0.0444$^{(9)}$	0.0404$^{(6)}$	0.0382$^{(5)}$	0.0569$^{(11)}$	0.0051$^{(1)}$	0.0247$^{(3)}$	0.0122$^{(2)}$	0.0353$^{(4)}$	0.0417$^{(8)}$	0.0694$^{(12)}$	0.0522$^{(10)}$
	ABB	α	1.4563$^{(5)}$	1.6412$^{(7)}$	1.6531$^{(8)}$	1.7032$^{(10)}$	1.6911$^{(9)}$	0.0123$^{(1)}$	1.3198$^{(3)}$	0.0454$^{(2)}$	1.4246$^{(4)}$	1.5915$^{(6)}$	1.8965$^{(12)}$	1.8586$^{(11)}$
		θ	0.2018$^{(7)}$	0.2107$^{(9)}$	0.2010$^{(6)}$	0.1954$^{(5)}$	0.2385$^{(11)}$	0.0718$^{(1)}$	0.1571$^{(3)}$	0.1104$^{(2)}$	0.1878$^{(4)}$	0.2043$^{(8)}$	0.2635$^{(12)}$	0.2285$^{(10)}$
	MRE	α	0.5394$^{(5)}$	0.6079$^{(7)}$	0.6123$^{(8)}$	0.6308$^{(10)}$	0.6263$^{(9)}$	0.0045$^{(1)}$	0.4888$^{(3)}$	0.0168$^{(2)}$	0.5276$^{(4)}$	0.5894$^{(6)}$	0.7024$^{(12)}$	0.6884$^{(11)}$
		θ	0.1442$^{(7)}$	0.1505$^{(9)}$	0.1436$^{(6)}$	0.1396$^{(5)}$	0.1703$^{(11)}$	0.0513$^{(1)}$	0.1122$^{(3)}$	0.0788$^{(2)}$	0.1341$^{(4)}$	0.1459$^{(8)}$	0.1882$^{(12)}$	0.1632$^{(10)}$
	Sum Rank		36$^{(5)}$	48$^{(9)}$	42$^{(6.5)}$	45$^{(8)}$	60$^{(10)}$	6$^{(1)}$	18$^{(3)}$	12$^{(2)}$	24$^{(4)}$	42$^{(6.5)}$	72$^{(12)}$	63$^{(11)}$

Table 6. The MSE, ABB and MRE values for ψ_{S4}.

n		Parameter	MLE	ADE	CvME	LSE	WLSE	KME	MPSE	MSADE	MSALDE	ADLTE	ADLTSOE	ADRTE
25	MSE	α	1.1438$^{(3)}$	1.1450$^{(4)}$	1.5918$^{(9)}$	1.8287$^{(10)}$	1.5047$^{(7)}$	0.0002$^{(1)}$	1.5172$^{(8)}$	0.1110$^{(2)}$	1.2001$^{(6)}$	1.8451$^{(11)}$	3.8929$^{(12)}$	1.1487$^{(5)}$
		θ	0.0013$^{(3)}$	0.0014$^{(4)}$	0.0018$^{(10)}$	0.0019$^{(11)}$	0.0017$^{(7)}$	0.0006$^{(1)}$	0.0016$^{(6)}$	0.0012$^{(2)}$	0.0016$^{(5)}$	0.0018$^{(8)}$	0.0025$^{(12)}$	0.0018$^{(9)}$
	ABB	α	1.0695$^{(3)}$	1.0701$^{(4)}$	1.2616$^{(9)}$	1.3523$^{(10)}$	1.2267$^{(7)}$	0.0149$^{(1)}$	1.2318$^{(8)}$	0.3332$^{(2)}$	1.0955$^{(6)}$	1.3584$^{(11)}$	1.9730$^{(12)}$	1.0718$^{(5)}$
		θ	0.0362$^{(3)}$	0.0380$^{(4)}$	0.0429$^{(10)}$	0.0440$^{(11)}$	0.0412$^{(7)}$	0.0251$^{(1)}$	0.0396$^{(6)}$	0.0349$^{(2)}$	0.0395$^{(5)}$	0.0420$^{(8)}$	0.0503$^{(12)}$	0.0420$^{(9)}$
	MRE	α	0.7130$^{(3)}$	0.7134$^{(4)}$	0.8411$^{(9)}$	0.9015$^{(10)}$	0.8178$^{(7)}$	0.0099$^{(1)}$	0.8212$^{(8)}$	0.2221$^{(2)}$	0.7303$^{(6)}$	0.9056$^{(11)}$	1.3154$^{(12)}$	0.7145$^{(5)}$
		θ	0.1205$^{(3)}$	0.1268$^{(4)}$	0.1432$^{(10)}$	0.1467$^{(11)}$	0.1374$^{(7)}$	0.0836$^{(1)}$	0.1321$^{(6)}$	0.1165$^{(2)}$	0.1318$^{(5)}$	0.1399$^{(8)}$	0.1677$^{(12)}$	0.1401$^{(9)}$
	Sum Rank		18$^{(3)}$	24$^{(4)}$	57$^{(9.5)}$	63$^{(11)}$	42$^{(7)}$	6$^{(1)}$	42$^{(7)}$	12$^{(2)}$	33$^{(5)}$	57$^{(9.5)}$	72$^{(12)}$	42$^{(7)}$
50	MSE	α	0.7652$^{(3)}$	0.8084$^{(4)}$	1.1396$^{(9)}$	1.2372$^{(10)}$	0.9652$^{(8)}$	0.0002$^{(1)}$	0.9029$^{(7)}$	0.0963$^{(2)}$	0.8268$^{(6)}$	1.2982$^{(11)}$	2.6595$^{(12)}$	0.8190$^{(5)}$
		θ	0.0009$^{(3)}$	0.0010$^{(4)}$	0.0013$^{(10)}$	0.0013$^{(11)}$	0.0011$^{(7)}$	0.0004$^{(1)}$	0.0010$^{(5)}$	0.0009$^{(2)}$	0.0011$^{(6)}$	0.0012$^{(9)}$	0.0019$^{(12)}$	0.0012$^{(8)}$
	ABB	α	0.8748$^{(3)}$	0.8991$^{(4)}$	1.0675$^{(9)}$	1.1123$^{(10)}$	0.9824$^{(8)}$	0.0146$^{(1)}$	0.9502$^{(7)}$	0.3103$^{(2)}$	0.9093$^{(6)}$	1.1394$^{(11)}$	1.6308$^{(12)}$	0.9050$^{(5)}$
		θ	0.0299$^{(3)}$	0.0313$^{(4)}$	0.0358$^{(10)}$	0.0364$^{(11)}$	0.0335$^{(7)}$	0.0210$^{(1)}$	0.0315$^{(5)}$	0.0298$^{(2)}$	0.0329$^{(6)}$	0.0352$^{(9)}$	0.0431$^{(12)}$	0.0348$^{(8)}$
	MRE	α	0.5832$^{(3)}$	0.5994$^{(4)}$	0.7117$^{(9)}$	0.7415$^{(10)}$	0.6550$^{(8)}$	0.0097$^{(1)}$	0.6335$^{(7)}$	0.2069$^{(2)}$	0.6062$^{(6)}$	0.7596$^{(11)}$	1.0872$^{(12)}$	0.6033$^{(5)}$
		θ	0.0997$^{(3)}$	0.1044$^{(4)}$	0.1192$^{(10)}$	0.1215$^{(11)}$	0.1116$^{(7)}$	0.0699$^{(1)}$	0.1051$^{(5)}$	0.0995$^{(2)}$	0.1098$^{(6)}$	0.1173$^{(9)}$	0.1437$^{(12)}$	0.1160$^{(8)}$
	Sum Rank		18$^{(3)}$	24$^{(4)}$	57$^{(9)}$	63$^{(11)}$	45$^{(8)}$	6$^{(1)}$	36$^{(5.5)}$	12$^{(2)}$	36$^{(5.5)}$	60$^{(10)}$	72$^{(12)}$	39$^{(7)}$
75	MSE	α	0.5867$^{(3)}$	0.6132$^{(4)}$	0.8595$^{(9)}$	0.9009$^{(10)}$	0.7019$^{(8)}$	0.0002$^{(1)}$	0.6385$^{(6)}$	0.0830$^{(2)}$	0.6330$^{(5)}$	1.0016$^{(11)}$	2.1031$^{(12)}$	0.6455$^{(7)}$
		θ	0.0007$^{(3)}$	0.0007$^{(5)}$	0.0010$^{(10)}$	0.0010$^{(11)}$	0.0008$^{(6)}$	0.0003$^{(1)}$	0.0007$^{(4)}$	0.0007$^{(2)}$	0.0008$^{(7)}$	0.0009$^{(9)}$	0.0015$^{(12)}$	0.0009$^{(8)}$
	ABB	α	0.7660$^{(3)}$	0.7831$^{(4)}$	0.9271$^{(9)}$	0.9491$^{(10)}$	0.8378$^{(8)}$	0.0142$^{(1)}$	0.7991$^{(6)}$	0.2880$^{(2)}$	0.7956$^{(5)}$	1.0008$^{(11)}$	1.4502$^{(12)}$	0.8035$^{(7)}$
		θ	0.0262$^{(3)}$	0.0273$^{(5)}$	0.0311$^{(10)}$	0.0315$^{(11)}$	0.0287$^{(6)}$	0.0185$^{(1)}$	0.0270$^{(4)}$	0.0262$^{(2)}$	0.0289$^{(7)}$	0.0306$^{(9)}$	0.0382$^{(12)}$	0.0304$^{(8)}$
	MRE	α	0.5106$^{(3)}$	0.5220$^{(4)}$	0.6181$^{(9)}$	0.6328$^{(10)}$	0.5585$^{(8)}$	0.0095$^{(1)}$	0.5327$^{(6)}$	0.1920$^{(2)}$	0.5304$^{(5)}$	0.6672$^{(11)}$	0.9668$^{(12)}$	0.5356$^{(7)}$
		θ	0.0873$^{(3)}$	0.0910$^{(5)}$	0.1037$^{(10)}$	0.1050$^{(11)}$	0.0957$^{(6)}$	0.0616$^{(1)}$	0.0900$^{(4)}$	0.0872$^{(2)}$	0.0962$^{(7)}$	0.1019$^{(9)}$	0.1272$^{(12)}$	0.1013$^{(8)}$
	Sum Rank		18$^{(3)}$	27$^{(4)}$	57$^{(9)}$	63$^{(11)}$	42$^{(7)}$	6$^{(1)}$	30$^{(5)}$	12$^{(2)}$	36$^{(6)}$	60$^{(10)}$	72$^{(12)}$	45$^{(8)}$
100	MSE	α	0.4702$^{(3)}$	0.5120$^{(5)}$	0.6975$^{(9)}$	0.7290$^{(10)}$	0.5630$^{(8)}$	0.0002$^{(1)}$	0.5000$^{(4)}$	0.0768$^{(2)}$	0.5154$^{(6)}$	0.8183$^{(11)}$	1.7601$^{(12)}$	0.5358$^{(7)}$
		θ	0.0006$^{(2)}$	0.0006$^{(5)}$	0.0008$^{(10)}$	0.0008$^{(11)}$	0.0007$^{(6)}$	0.0003$^{(1)}$	0.0006$^{(4)}$	0.0006$^{(3)}$	0.0007$^{(7)}$	0.0007$^{(9)}$	0.0012$^{(12)}$	0.0007$^{(8)}$
	ABB	α	0.6857$^{(3)}$	0.7155$^{(5)}$	0.8352$^{(9)}$	0.8538$^{(10)}$	0.7503$^{(8)}$	0.0140$^{(1)}$	0.7071$^{(4)}$	0.2772$^{(2)}$	0.7179$^{(6)}$	0.9046$^{(11)}$	1.3267$^{(12)}$	0.7320$^{(7)}$
		θ	0.0236$^{(2)}$	0.0244$^{(5)}$	0.0279$^{(10)}$	0.0282$^{(11)}$	0.0255$^{(6)}$	0.0168$^{(1)}$	0.0238$^{(4)}$	0.0238$^{(3)}$	0.0258$^{(7)}$	0.0273$^{(9)}$	0.0346$^{(12)}$	0.0272$^{(8)}$
	MRE	α	0.4572$^{(3)}$	0.4770$^{(5)}$	0.5568$^{(9)}$	0.5692$^{(10)}$	0.5002$^{(8)}$	0.0093$^{(1)}$	0.4714$^{(4)}$	0.1848$^{(2)}$	0.4786$^{(6)}$	0.6031$^{(11)}$	0.8844$^{(12)}$	0.4880$^{(7)}$
		θ	0.0786$^{(2)}$	0.0815$^{(5)}$	0.0932$^{(10)}$	0.0940$^{(11)}$	0.0851$^{(6)}$	0.0561$^{(1)}$	0.0794$^{(4)}$	0.0794$^{(3)}$	0.0861$^{(7)}$	0.0910$^{(9)}$	0.1153$^{(12)}$	0.0908$^{(8)}$
	Sum Rank		15$^{(2.5)}$	30$^{(5)}$	57$^{(9)}$	63$^{(11)}$	42$^{(7)}$	6$^{(1)}$	24$^{(4)}$	15$^{(2.5)}$	39$^{(6)}$	60$^{(10)}$	72$^{(12)}$	45$^{(8)}$
250	MSE	α	0.3332$^{(3)}$	0.3688$^{(5)}$	0.4991$^{(9)}$	0.5178$^{(10)}$	0.3980$^{(8)}$	0.0002$^{(1)}$	0.3419$^{(4)}$	0.0632$^{(2)}$	0.3709$^{(6)}$	0.5805$^{(11)}$	1.2878$^{(12)}$	0.3878$^{(7)}$
		θ	0.0004$^{(3)}$	0.0004$^{(5)}$	0.0005$^{(10)}$	0.0006$^{(11)}$	0.0005$^{(6)}$	0.0002$^{(1)}$	0.0004$^{(2)}$	0.0004$^{(4)}$	0.0005$^{(7)}$	0.0005$^{(9)}$	0.0009$^{(12)}$	0.0005$^{(8)}$
	ABB	α	0.5773$^{(3)}$	0.6073$^{(5)}$	0.7065$^{(9)}$	0.7196$^{(10)}$	0.6309$^{(8)}$	0.0137$^{(1)}$	0.5847$^{(4)}$	0.2514$^{(2)}$	0.6090$^{(6)}$	0.7619$^{(11)}$	1.1348$^{(12)}$	0.6227$^{(7)}$
		θ	0.0200$^{(3)}$	0.0207$^{(5)}$	0.0234$^{(10)}$	0.0236$^{(11)}$	0.0213$^{(6)}$	0.0144$^{(1)}$	0.0199$^{(2)}$	0.0204$^{(4)}$	0.0219$^{(7)}$	0.0230$^{(9)}$	0.0297$^{(12)}$	0.0229$^{(8)}$
	MRE	α	0.3848$^{(3)}$	0.4049$^{(5)}$	0.4710$^{(9)}$	0.4797$^{(10)}$	0.4206$^{(8)}$	0.0092$^{(1)}$	0.3898$^{(4)}$	0.1676$^{(2)}$	0.4060$^{(6)}$	0.5079$^{(11)}$	0.7565$^{(12)}$	0.4152$^{(7)}$
		θ	0.0667$^{(3)}$	0.0689$^{(5)}$	0.0782$^{(10)}$	0.0788$^{(11)}$	0.0711$^{(6)}$	0.0481$^{(1)}$	0.0664$^{(2)}$	0.0680$^{(4)}$	0.0730$^{(7)}$	0.0766$^{(9)}$	0.0989$^{(12)}$	0.0764$^{(8)}$
	Sum Rank		18$^{(3)}$	30$^{(5)}$	57$^{(9)}$	63$^{(11)}$	42$^{(7)}$	6$^{(1)}$	18$^{(3)}$	18$^{(3)}$	39$^{(6)}$	60$^{(10)}$	72$^{(12)}$	45$^{(8)}$
500	MSE	α	0.2345$^{(4)}$	0.2597$^{(5)}$	0.3476$^{(9)}$	0.3576$^{(10)}$	0.2777$^{(8)}$	0.0002$^{(1)}$	0.2308$^{(3)}$	0.0519$^{(2)}$	0.2651$^{(6)}$	0.4084$^{(11)}$	0.9483$^{(12)}$	0.2740$^{(7)}$
		θ	0.0003$^{(3)}$	0.0003$^{(4)}$	0.0004$^{(10)}$	0.0004$^{(11)}$	0.0003$^{(6)}$	0.0001$^{(1)}$	0.0003$^{(2)}$	0.0003$^{(5)}$	0.0003$^{(7)}$	0.0004$^{(9)}$	0.0006$^{(12)}$	0.0004$^{(8)}$
	ABB	α	0.4843$^{(4)}$	0.5097$^{(5)}$	0.5896$^{(9)}$	0.5980$^{(10)}$	0.5270$^{(8)}$	0.0135$^{(1)}$	0.4804$^{(3)}$	0.2277$^{(2)}$	0.5148$^{(6)}$	0.6391$^{(11)}$	0.9738$^{(12)}$	0.5234$^{(7)}$
		θ	0.0167$^{(3)}$	0.0173$^{(4)}$	0.0195$^{(10)}$	0.0196$^{(11)}$	0.0178$^{(6)}$	0.0121$^{(1)}$	0.0164$^{(2)}$	0.0173$^{(5)}$	0.0184$^{(7)}$	0.0192$^{(9)}$	0.0255$^{(12)}$	0.0192$^{(8)}$
	MRE	α	0.3228$^{(4)}$	0.3398$^{(5)}$	0.3931$^{(9)}$	0.3987$^{(10)}$	0.3513$^{(8)}$	0.0090$^{(1)}$	0.3203$^{(3)}$	0.1518$^{(2)}$	0.3432$^{(6)}$	0.4261$^{(11)}$	0.6492$^{(12)}$	0.3489$^{(7)}$
		θ	0.0558$^{(3)}$	0.0575$^{(4)}$	0.0649$^{(10)}$	0.0655$^{(11)}$	0.0593$^{(6)}$	0.0403$^{(1)}$	0.0548$^{(2)}$	0.0577$^{(5)}$	0.0614$^{(7)}$	0.0641$^{(9)}$	0.0849$^{(12)}$	0.0640$^{(8)}$
	Sum Rank		21$^{(3.5)}$	27$^{(5)}$	57$^{(9)}$	63$^{(11)}$	42$^{(7)}$	6$^{(1)}$	15$^{(2)}$	21$^{(3.5)}$	39$^{(6)}$	60$^{(10)}$	72$^{(12)}$	45$^{(8)}$

Table 7. Rank and rank score of MSE, ABB, and MRE for the ψ.

Scenarios	n	MLE	ADE	CvME	LSE	WLSE	KME	MPSE	MSADE	MSALDE	ADLTE	ADLTSOE	ADRTE
ψ_{S1}	**25**	3	2	9.5	11	6	1	7	5	8	9.5	12	4
	50	2	3	9.5	11	6.5	1	6.5	4.5	8	9.5	12	4.5
	75	2	3	8.5	11	5.5	1	5.5	7	8.5	10	12	4
	100	2	3	8.5	11	6	1	5	7	8.5	10	12	4
	250	2	3	8	11	6	1	5	7	9.5	9.5	12	4
	500	2	4	9.5	11	6	1	3	8	7	9.5	12	5
ψ_{S2}	**25**	3	4	5	11	7.5	1	10	2	7.5	6	12	9
	50	2.5	4	7	10.5	5	1	8	2.5	10.5	6	12	9
	75	2	3.5	7	10	5	1	8	3.5	10	6	12	10
	100	2	3	8	9	4.5	1	6.5	4.5	11	6.5	12	10
	250	2	3	8	9	4	1	5.5	5.5	11	7	12	10
	500	2	3	8	9	4	1	5	6	10.5	7	12	10.5
ψ_{S3}	**25**	7.5	9	5.5	5.5	11.5	1	4	2	3	7.5	11.5	10
	50	7.5	9	6	5	11	1	3.5	2	3.5	7.5	12	10
	75	8	9	5	6	11	1	3.5	2	3.5	7	12	10
	100	7.5	9	5.5	5.5	10.5	1	3	2	4	7.5	12	10.5
	250	6	9	6	8	10.5	1	3	2	4	6	12	10.5
	500	5	9	6.5	8	10	1	3	2	4	6.5	12	11
ψ_{S4}	**25**	3	4	9.5	11	7	1	7	2	5	9.5	12	7
	50	3	4	9	11	8	1	5.5	2	5.5	10	12	7
	75	3	4	9	11	7	1	5	2	6	10	12	8
	100	2.5	5	9	11	7	1	4	2.5	6	10	12	8
	250	3	5	9	11	7	1	3	3	6	10	12	8
	500	3.5	5	9	11	7	1	2	3.5	6	10	12	8
Sum Rank		86	119.5	185.5	228.5	173.5	24	121.5	89.5	166.5	198	287.5	192
Overall Rank		2	4	8	11	7	1	5	3	6	10	12	9

5. PRACTICAL EXAMPLES

In this section, two real-data analyses are conducted to demonstrate the applicability of the proposed distribution. The UTPA distribution is compared to the B, KW [2], UT [39], UW [7] and UB-XII [40] distributions. The $\hat{\ell}$, Akaike's information criterion (AIC), Bayesian information criterion (BIC), consistent AIC (CAIC), Hannan-Quinn information criterion (HQIC), standard error (se) of ML estimate and Kolmogorov-Smirnov (KS) statistics and KS p-values are calculated for both data sets. The pdfs of B, KW, UT, UW, and UB-XII are given in Equations (19)-(23), respectively.

$$f_B\left(x;p_1,p_2\right)=\frac{\Gamma\left(p_1+p_2\right)}{\Gamma\left(p_1\right)\Gamma\left(p_2\right)}x^{\left(p_1-1\right)}\left(1-x\right)^{\left(p_2-1\right)}, \tag{19}$$

$$f_{KW}\left(x;p_1,p_2\right)=\left(1-x^{p_1}\right)^{\left(p_2-1\right)}p_2x^{\left(p_1-1\right)}p_1, \tag{20}$$

$$f_{UT}\left(x;p_1\right)=p_1\left(x^{-p_1}-1\right)x^{-\left(p_1+1\right)}\exp\left(-x^{-p_1+1}\right), \tag{21}$$

$$f_{UW}\left(x;p_1,p_2\right)=x^{(-1)}p_1p_2\left(-\log\left(x\right)\right)^{\left(p_2-1\right)}\exp\left(-p_1\left(-\log\left(x\right)\right)^{p_2}\right), \tag{22}$$

and

$$f_{UB-XII}\left(x;p_1,p_2\right)=x^{(-1)}p_1p_2\left(-\log\left(x\right)\right)^{\left(p_2-1\right)}\left(1+\left(-\log\left(x\right)\right)^{p_2}\right)^{\left(-p_1-1\right)}, \tag{23}$$

where $p_1, p_2>0$, and $0<x<1$.

5.1. REAL DATA EXAMPLE 1

The first data set contains information on a right-skewed dataset discussed by [41], representing the lifespans of T8 fluorescent lamps for 50 devices, measured in hours. The data are: 0.445, 0.493, 0.285, 0.564, 0.76, 0.381, 0.69, 0.579, 0.636, 0.238, 0.149, 0.244, 0.126, 0.796, 0.405, 0.553, 0.78, 0.431, 0.184, 0.375, 0.198, 0.89, 0.192, 0.463, 0.486, 0.521, 0.366, 0.486, 0.116, 0.511, 0.612, 0.117, 0.384, 0.326, 0.057, 0.412, 0.586, 0.517, 0.57, 0.588, 0.497, 0.246, 0.234, 0.228, 0.552, 0.893, 0.403, 0.458, 0.134, 0.338. The goodness-of-fit results for the real data are systematically summarized in Table 8, where the proposed UTPA distribution is compared with several existing models from the literature using multiple model selection and fit criteria. As shown in the table, the UTPA distribution consistently achieves the lowest AIC, BIC, CAIC, and HQIC values among all competing models, indicating the best model fit. In addition, it yields the smallest KS statistic and the largest p-value, further supporting its adequacy in capturing the underlying distributional structure of the data. Furthermore, the fit to the UTPA distribution for to-right data is shown in Fig. 4, with fitted pdf, cdf, sf, and probability–probability (P–P) plots.

Table 8. Data analysis results for the first real data.

Model	UTPA	B	KW	UT	UW	UB-XII
$\hat{\ell}$	10.1049	10.0291	9.8845	9.4142	9.9141	9.2374
AIC	-16.2098	-16.0582	-15.7690	-16.8284	-15.8281	-14.4747
BIC	-12.3857	-12.2341	-11.9449	-14.9164	-12.0041	-10.6507
CAIC	-15.9544	-15.8028	-15.5137	-16.7451	-15.5728	-14.2194
HQIC	-14.7535	-14.6019	-14.3128	-16.1003	-14.3719	-13.0185
KS	0.0756	0.0837	0.0878	0.1268	0.0950	0.0868
p-value	0.9375	0.8748	0.8354	0.3972	0.7578	0.8457
p_1	0.0510	2.7179	2.8940		1.7404	2.4224
p_2	2.8763	2.0664	1.8098	1.8172	0.8251	1.3879
se for p_1	0.0790	0.5280	0.6549		0.1887	0.2883
se for p_2	0.3166	0.3903	0.2628	0.2570	0.1297	0.2018

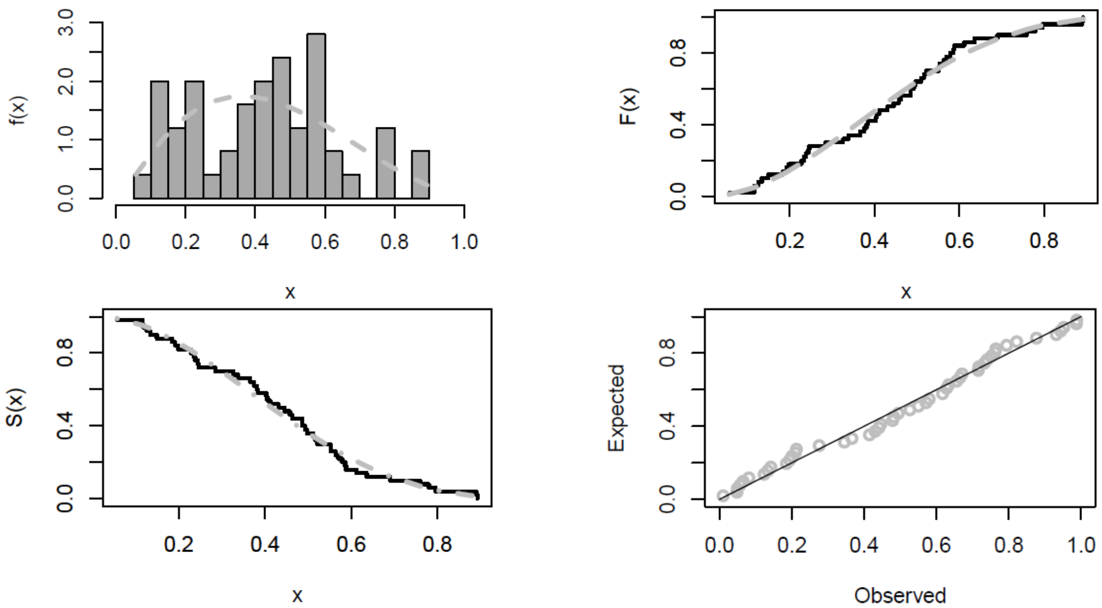

Figure 4. The ftted pdf, cdf, sf, and P–P plots for the UTPA distribution of the to-right data.

5.2. REAL DATA EXAMPLE 2

The second data set, following the skewed-to-the-right data, represents two different load duration curves referred to as SC16 and P3. For more details, see [42]. The data consists of 23 samples. The data are: 0.853, 0.759, 0.866, 0.809, 0.717, 0.544, 0.492, 0.403, 0.344, 0.213, 0.116, 0.116, 0.092, 0.070, 0.059, 0.048, 0.036, 0.029, 0.021, 0.014, 0.011, 0.008, 0.006. The goodness-of-fit results for the real data are systematically presented in Table 9, where the proposed UTPA distribution is compared with several well-established models from the literature using multiple model selection and goodness-of-fit criteria. As shown in Table 9, the UTPA distribution achieves the lowest AIC, BIC, CAIC, and HQIC among all competing models, indicating it provides the best overall fit to the data. Furthermore, by yielding the smallest KS statistic and the largest p-value, the proposed distribution also confirms its effectiveness by providing the best fit for the second dataset. In addition to the numerical comparisons, the graphical assessments displayed in Fig. 5 demonstrate the strong agreement between the UTPA model and the right-skewed real data.

Table 9. Data analysis results for the second real data.

Model	UTPA	B	KW	UT	UW	UB-XII
$\hat{\ell}$	10.1583	9.6075	9.6708	8.1151	9.9631	5.8728
AIC	-16.3167	-15.2149	-15.3416	-14.2302	-15.9262	-7.7456
BIC	-14.0457	-12.9439	-13.0706	-13.0948	-13.6553	-5.4746
CAIC	-15.7167	-14.6149	-14.7416	-14.0398	-15.3262	-7.1456
HQIC	-15.7455	-14.6438	-14.7704	-13.9447	-15.3551	-7.1744
KS	0.1563	0.1836	0.1790	0.1690	0.1636	0.2558
p-value	0.6276	0.4202	0.4529	0.5272	0.5697	0.0985
p_1	2.2050	1.1679	1.1862		1.1952	1.5050
p_2	0.8380	0.4869	0.5044	0.5943	0.3645	0.7848
se for p_1	2.1869	0.3578	0.3265		0.2106	0.2951
se for p_2	0.2262	0.1208	0.1288	0.1239	0.1169	0.1919

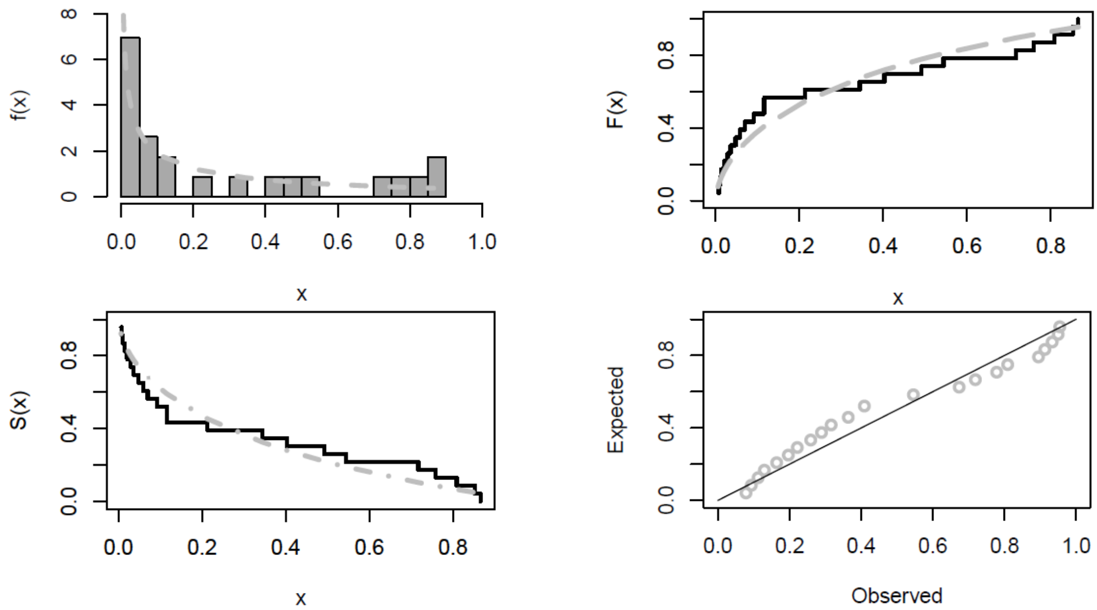

Figure 5. The ftted pdf, cdf, sf, and P–P plots for the UTPA distribution of the skewed-to-right data.

6. CONCLUSIONS

In this study, a new distribution is proposed to model data on the unit interval. Several properties of the proposed distribution are examined, including moments, skewness and kurtosis coefficients, Bonferroni and Lorenz curves and order statistics. To evaluate the parameter estimations, a Monte Carlo simulation study is conducted using twelve different estimators. According to the simulation results, the best estimators across the three criteria are KME, MLE, MSADE, and ADE, while the weakest are ADLTSOE, LSE, ADLTE, and ADRTE. Furthermore, two real datasets demonstrate that the proposed distribution provides a better fit than competitor distributions. The motivation for this study stems from the observation that the proposed distribution exhibits superior goodness of fit across two real datasets compared to competing distributions. Future studies could focus on extending the proposed distribution to a regression framework to model bounded response variables as functions of covariates. Additionally, developing a bivariate version of this distribution would enable analysis of interdependent unit-interval data. Finally, investigating the model's performance under censored data remains an important area for further exploration to enhance its applicability in survival analysis.

REFERENCES

[1] Topp, C. W., Leone, F. C., *Journal of the American Statistical Association*, **50**(269), 209, 1955.
[2] Kumaraswamy, P., *Journal of Hydrology*, **46**(1-2), 79, 1980.
[3] Mazucheli, J., Menezes, A. F., Dey, S., *Communications in Statistics-Theory and Methods*, **47**(15), 3767, 2018.
[4] Mazucheli, J., Menezes, A. F., Dey, S., *Chilean Journal of Statistics*, **9**(1), 47, 2018.
[5] Menezes, A. F. B., Mazucheli, J., Dey, S., *Pesquisa Operacional*, **38**(3), 555, 2018.
[6] Ghitany, M., Mazucheli, J., Menezes, A., Alqallaf, F., *Communications in Statistics-Theory and Methods*, **48**(14), 3423, 2019.
[7] Mazucheli, J., Menezes, A., Fernandes, L., De Oliveira, R., Ghitany, M., *Journal of Applied Statistics*, **47**(6), 954, 2020.
[8] Sindhu, T. N., Shafiq, A., Riaz, M. B., Abushal, T. A., *Heliyon*, 2024.
[9] Altun, E., Hamedani, G., Fazli, A., *Cumhuriyet Science Journal*, **45**(4), 803, 2024.
[10] Al-Omari, A. I., Alanzi, A. R., Alshqaq, S. S., *Alexandria Engineering Journal*, **92**, 238, 2024.
[11] Alzahrani, M. R., Almohaimeed, M., *Alexandria Engineering Journal*, **117**, 193, 2025.
[12] Irshad, M. R., Aswathy, S., Maya, R., Al-Omari, A. I., Alomani, G., *AIMS Mathematics*, **9**(9), 24810, 2024.
[13] Stojanović, V. S., Jovanović Spasojević, T., Jovanović, M., *Mathematics*, **12**(14), 2282, 2024.
[14] Stojanović, V. S., Jovanović Spasojević, T., Bojičić, R., Pažun, B., Langović, Z., *Mathematics*, **13**(2), 255, 2025.
[15] Stojanović, V. S., Jovanović, M., Pažun, B., Langović, Z., Grujčić, Ž., *Symmetry*, **16**(11), 1513, 2024.
[16] Semary, H. E., Chesneau, C., Aldahlan, M. A., Elbatal, I., Elgarhy, M., Abdelwahab, M. M., Almetwally, E. M., *Alexandria Engineering Journal*, **100**, 340, 2024.
[17] Bashiru, S. O., Kayid, M., Mahmoud, R., Balogun, O. S., Abd El-Raouf, M. M., Gemeay, A. M., *Journal of Radiation Research and Applied Sciences*, **18**(1), 101204, 2025.
[18] Biçer, C., Bakouch, H. S., Biçer, H. D., Alomair, G., Hussain, T., Almohisen, A., *Axioms*, **13**(4), 226, 2024.
[19] Ribeiro-Reis, L. D., "The Unit Inverse Weibull Distribution and its Associated Regression Model", 2022.
[20] Haq, M. A. U., Hashmi, S., Aidi, K., Ramos, P. L., Louzada, F., *Annals of Data Science*, **10**(2), 415, 2020.
[21] Haj Ahmad, H., Almetwally, E. M., Elgarhy, M., Ramadan, D. A., *Processes*, **11**(1), 232, 2023.
[22] Muhammad, M., Abba, B., Xiao, J., Alsadat, N., Jamal, F., Elgarhy, M., *IEEE Access*, **12**, 156235, 2024.
[23] Hashmi, S., Ahsan-ul-Haq, M., Zafar, J., Khaleel, M. A., *Physical and Computational Sciences*, **59**(1), 15, 2022.
[24] Maya, R., Jodra, P., Irshad, M. R., Krishna, A., *Ricerche di Matematica*, **73**(4), 1843, 2024.

[25] Hassan, A. S., Fayomi, A., Algarni, A., Almetwally, E. M., *Applied Sciences*, **12**(21), 11253, 2022.
[26] Krishna, A., Maya, R., Chesneau, C., Irshad, M. R., *Mathematical and Computational Applications*, **27**(1), 12, 2022.
[27] Ramadan, A. T., Tolba, A. H., El-Desouky, B. S., *Axioms*, **11**(12), 676, 2022.
[28] Korkmaz, M. Ç., Korkmaz, Z. S., *Journal of Applied Statistics*, **50**(4), 889, 2023.
[29] Bashiru, S. O., Kayid, M., Sayed, R. M., Balogun, O. S., Abd El-Raouf, M. M., Gemeay, A. M., *Journal of Radiation Research and Applied Sciences*, **18**(1), 101204, 2025.
[30] Gemeay, A. M., Alsadat, N., Chesneau, C., Elgarhy, M., *AIMS Mathematics*, **9**(8), 20976, 2024.
[31] Shanker, R., Soni, N. K., Reliability: *Theory & Applications*, **19**(3 (79)), 757, 2024.
[32] Shaked, M., Shanthikumar, J. G., *Stochastic Orders*, Springer New York, 2007.
[33] Casella, G., Robert, C. P., Wells, M. T., *Lecture Notes-Monograph Series*, **45**, 342, 2004.
[34] Anderson, T. W., Darling, D. A., *The Annals of Mathematical Statistics*, **23**, 193, 1952.
[35] Choi, K., Bulgren, W. G., *Journal of the Royal Statistical Society Series B: Statistical Methodology*, **30**(3), 444, 1968.
[36] Swain, J. J., Venkatraman, S., Wilson, J. R., *Journal of Statistical Computation and Simulation*, **29**(4), 271, 1988.
[37] Cheng, R. C. H., Amin, N. A. K., *Journal of the Royal Statistical Society: Series B (Methodological)*, **45**(3), 394, 1983.
[38] Kuş, C., Eryılmaz, S., *Quality Technology & Quantitative Management*, **18**(6), 771, 2021.
[39] Krishna, A., Maya, R., Chesneau, C., Irshad, M. R., *Mathematical and Computational Applications*, **27**(1), 12, 2022.
[40] Korkmaz, M. Ç., Chesneau, C., *Computational and Applied Mathematics*, **40**(1), 29, 2021.
[41] Ahmed, M. A., *Colombian Journal of Statistics/Revista Colombiana de Estadística*, **43**(2), 285, 2020.
[42] Caramanis, M., Stremel, J., Fleck, W., Daniel, S., *International Journal of Electrical Power & Energy Systems*, **5**(2), 75, 1983.