***ORIGINAL PAPER***

# ON AXIAL WRAPPED EXPONENTIAL DISTRIBUTION WITH ESTIMATION AND APPLICATION

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***Abstract.*** *This paper applies a wrapped exponential distribution to the axial domain for arcs of arbitrary lengths. Furthermore, we provide a closed-form expression for the characteristic function of the l-axial wrapped exponential distribution. We conduct a comprehensive analysis of population characteristics, parameter estimation techniques, and simulation studies, focusing primarily on the semicircular exponential distribution. Additionally, we discuss the practical applicability of this distribution to real-world datasets.*

***Keywords:*** *Axial data; characteristic function; circular model; semicircular model; trigonometric moments.*

***Mathematics Subject Classification:*** *62H11; 60E05.*

## 1. INTRODUCTION

Jammalamadaka and Kozubowski [1] initially developed the wrapped exponential distribution in 2000. Levy [2] first introduced the concept of wrapped distribution in 1939, but numerous authors, including Mardia et al. [3], Jammalamadaka et al. [4-5], Coelho [6], Rao et al. [7], Roy et al. [8], Joshi et al. [9], Abdullah et al. [10], Bhattacharjee [11], Al-Khazaleh et al. [12], Chesneau et al. [13], Zinhom et al. [14], and Phani et al. [15-16] have derived various wrapped distributions. These distributions encompass a variety of models, including wrapped exponential, wrapped Laplace, wrapped logistic, wrapped extreme-value, wrapped gamma, wrapped weighted exponential, wrapped Lindley, transmuted wrapped exponential, wrapped length-biased weighted exponential, wrapped Akash, wrapped modified Lindley, wrapped XLindley, wrapped length-biased exponential, and wrapped size-biased Lindley distributions, respectively. Jones et al. [17] proposed a novel family of symmetric unimodal distributions on the unit circle, including circular uniform, cardioid, von Mises, and wrapped Cauchy models as special cases. Several authors, including Ugai et al. [18], Guardiola [19], Hyoung-Moon Kim et al. [20-21], Girija et al. [22], Phani et al. [23-25], Ali H. Abuzaid [26], Ayesha et al. [27], and Radhika et al. [28], have observed that while these models are suitable for circular data analysis, they are not necessary for dealing with axial data. Consequently, researchers have recently developed several axial and semicircular models and investigated their applicability to real-world scenarios.

The objective of the present study is to derive the l-axial exponential distribution in a general context and conduct a comprehensive examination of the semicircular exponential distribution in particular. We structure the remainder of the article as follows: In Section 2, we show how the l-axial wrapped exponential distribution and its submodel, the semicircular

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exponential distribution, are derived. Section 3 discusses characteristic functions, trigonometric moments, and population characteristics. In Section 4, we apply the maximum-likelihood method for parameter estimation and conduct a simulation study to assess its consistency and efficiency. Finally, to evaluate the adequacy of the proposed model, we analyze a real data set that has measurements of the long-axis orientation of 133 feldspar laths [29]. We then compare its performance with other models available in the literature.

## 2. $l-$ AXIAL WRAPPED EXPONENTIAL DISTRIBUTION

Jammalamadaka et al. [1] derived a one-parameter wrapped model by wrapping a classical exponential model around a unit circle and derived some statistical properties. In this study, we generalized the model to a family of axial models.

**Definition 2.1.** A random variable $X_{laW}$ is said to have the $l-$axial wrapped exponential distribution with scale parameter $\lambda > 0$, denoted by $l-$ AWE$(\lambda)$, if the probability density and distribution functions are respectively given by

$$g(\phi) = \frac{l\,\lambda \exp(-l\lambda\phi)}{1-\exp(-2\pi\lambda)}, \tag{1}$$

and $G(\phi) = \dfrac{1-\exp(-l\lambda\phi)}{1-\exp(-2\pi\lambda)}$ , where

$$\lambda > 0\,, 0 < \phi \le \frac{2\pi}{l}\ ,\ l = 1, 2, 3, ...,\ . \tag{2}$$

***Special Cases:***

(i) When $l = 1$ in (1), we get the density function of the wrapped exponential distribution.

(ii) When $l = 2$ in (1), we get the density function of $2-$axial-wrapped exponential distribution; we call it the semicircular-wrapped exponential distribution.

(iii) When $l = 3$ in (1), we get the density function of $3-$axial wrapped exponential distribution.

### Semicircular Wrapped Exponential Distribution

A random variable $X_{SC}$ has a semicircular wrapped exponential distribution if its probability density function and cumulative distribution function are given by

$$g(\theta) = \frac{2\,\lambda \exp(-2\lambda\theta)}{1-\exp(-2\pi\lambda)}\ \text{, where } \lambda > 0\,, 0 < \theta \le \pi\,. \tag{3}$$

$$G(\theta) = \frac{1-\exp(-2\lambda\theta)}{1-\exp(-2\pi\lambda)}\ \text{, where } \lambda > 0,\ 0 < \theta \le \pi. \tag{4}$$

Plots of the density and distribution function are presented below for various values of the parameter (Figs 1-4).

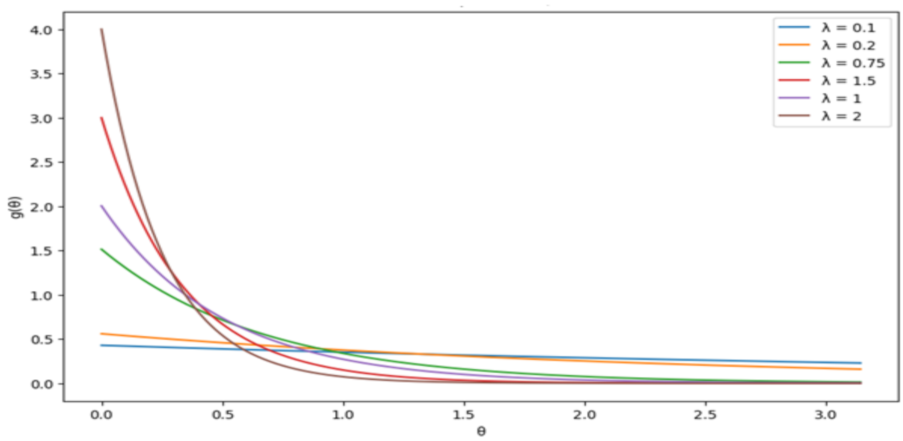


**Figure 1. Plots of the probability density function of the semicircular wrapped exponential distribution for various values of the parameter.**

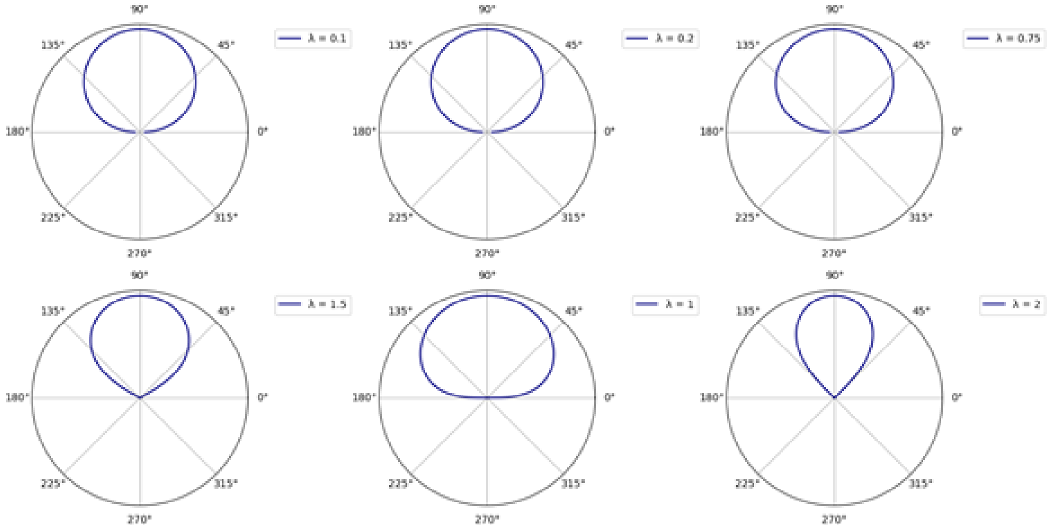


**Figure 2. Plots of probability density function (circular representation) of semicircular wrapped exponential distribution for various parameter values.**

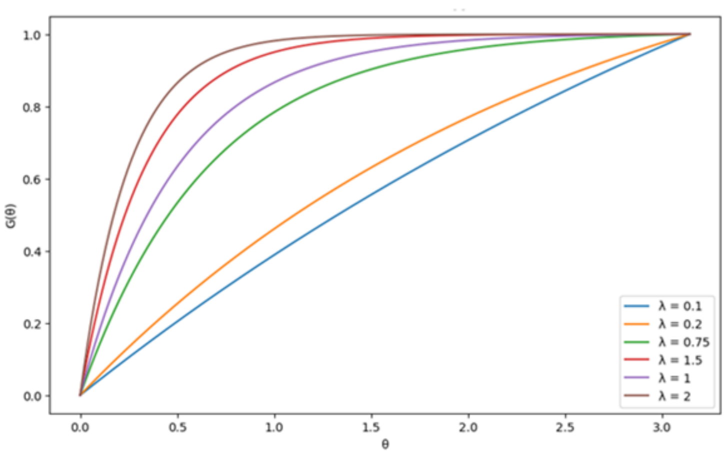


**Figure 3. Plots of the distribution function (linear representation) of the semicircular wrapped exponential distribution for various parameter values.**

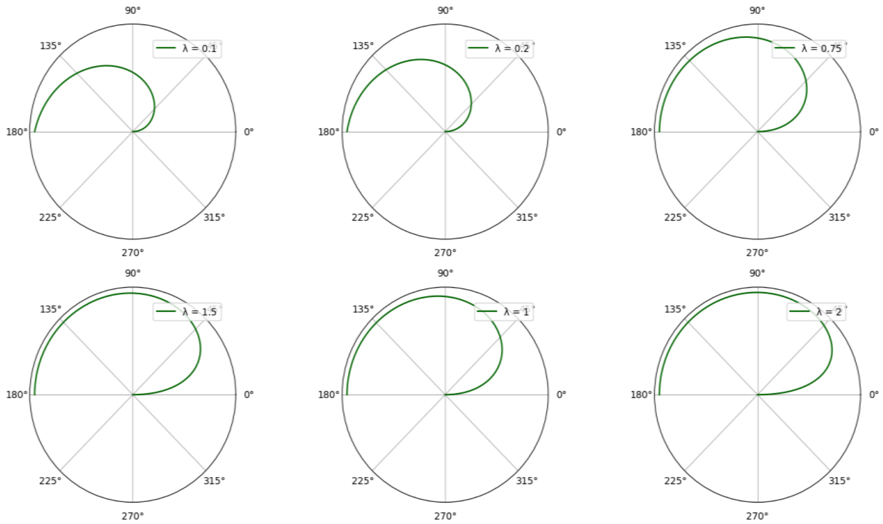


**Figure 4. Plots of the distribution function (circular representation) of the semicircular wrapped exponential distribution for various parameter values.**

## 3. CHARACTERISTIC FUNCTION OF $l-$ AXIAL WRAPPED EXPONENTIAL DISTRIBUTION

The characteristic function of a circular model with pdf $g(\theta)$ is given by

$$\varphi_\theta(p) = E\left(e^{ip\theta}\right) = \int_0^{2\pi} e^{ip\theta} g(\theta) d\theta ,$$

where

$$p \in Z .$$

$$\varphi_\theta(p) = \frac{l\lambda}{1-e^{-2\pi\lambda}} \int_0^{\frac{2\pi}{l}} e^{ip\theta} e^{-l\lambda\theta} d\theta$$

$$= \frac{l\lambda}{1-e^{-2\pi\lambda}} \int_0^{\frac{2\pi}{l}} e^{-(l\lambda - ip)\theta} d\theta ,$$

where

$$l = 1, 2, 3, \ldots,$$

$$= \frac{l\lambda}{1-e^{-2\pi\lambda}} \left( \frac{e^{-(l\lambda - ip)\theta}}{-(l\lambda - ip)} \right)_0^{\frac{2\pi}{l}}$$

$$\varphi_\theta(p) = \frac{l\lambda}{\left(1-e^{-2\pi\lambda}\right)\left(\left(l\lambda - ip\right)\right)} \left(1 - e^{-2\pi\lambda} \left( \cos\left(\frac{2\pi p}{l}\right) - i \sin\left(\frac{2\pi p}{l}\right)\right)\right). \quad (5)$$

### *3.1. CHARACTERISTIC OF SEMICIRCULAR WRAPPED EXPONENTIAL DISTRIBUTION*

In this section, we derive the population characteristics of the semicircular wrapped exponential distribution.

(1) Trigonometric moments

$$\alpha_p = \begin{cases} \dfrac{4\lambda^2}{4\lambda^2+p^2}\coth(\pi\lambda), \text{ if } p \text{ is odd} \\ \dfrac{4\lambda^2}{4\lambda^2+p^2}, \text{ if } p \text{ is even} \end{cases} \quad \text{and} \quad \beta_p = \begin{cases} \dfrac{2\lambda p}{4\lambda^2+p^2}\coth(\pi\lambda), \text{ if } p \text{ is odd} \\ \dfrac{2\lambda p}{4\lambda^2+p^2}, \text{ if } p \text{ is even} \end{cases}$$

(2) Central trigonometric moments

$$\alpha_p^* = \rho_p \cos\left(\mu_p^0 - p\mu_0\right) = \begin{cases} \dfrac{2\lambda}{\sqrt{4\lambda^2+p^2}}\cos\left(\tan^{-1}\left(\dfrac{p}{2\lambda}\right) - p\tan^{-1}\left(\dfrac{1}{2\lambda}\right)\right), \text{if } p \text{ is odd} \\ \dfrac{2\lambda}{\sqrt{4\lambda^2+p^2}}\tanh(\pi\lambda)\cos\left(\tan^{-1}\left(\dfrac{p}{2\lambda}\right) - p\tan^{-1}\left(\dfrac{1}{2\lambda}\right)\right), \text{if } p \text{ is even} \end{cases}$$

$$\beta_p^* = \rho_p \sin\left(\mu_p^0 - p\mu_0\right) = \begin{cases} \dfrac{2\lambda}{\sqrt{4\lambda^2+p^2}}\sin\left(\tan^{-1}\left(\dfrac{p}{2\lambda}\right) - p\tan^{-1}\left(\dfrac{1}{2\lambda}\right)\right), \text{if } p \text{ is odd} \\ \dfrac{2\lambda}{\sqrt{4\lambda^2+p^2}}\tanh(\pi\lambda)\sin\left(\tan^{-1}\left(\dfrac{p}{2\lambda}\right) - p\tan^{-1}\left(\dfrac{1}{2\lambda}\right)\right), \text{if } p \text{ is even} \end{cases}$$

where

$$\rho_p = \sqrt{\alpha_p^2 + \beta_p^2} = \begin{cases} \dfrac{2\lambda}{\sqrt{4\lambda^2+p^2}}, \text{if } p \text{ is odd} \\ \dfrac{2\lambda}{\sqrt{4\lambda^2+p^2}}\tanh(\pi\lambda), \text{if } p \text{ is even} \end{cases}$$

(1) Resultant length: $\rho = \rho_1 = \sqrt{\alpha_1^2 + \beta_1^2} = \dfrac{2\lambda}{\sqrt{4\lambda^2+1}}$.

(2) Mean direction: $\mu_0 = \mu_1^0 = \tan^{-1}\left(\dfrac{\beta_1}{\alpha_1}\right) = \tan^{-1}\left(\dfrac{1}{2\lambda}\right)$ .

(3) Circular variance: $V_0 = 1 - \rho = 1 - \rho_0 = \dfrac{\sqrt{4\lambda^2+1} - 2\lambda}{\sqrt{4\lambda^2+1}}$ .

(4) Circular standard deviation: $\sigma_0 = \sqrt{-2ln(1-V_0)} = \sqrt{ln\left(1 + \dfrac{1}{4\lambda^2}\right)}$ .

(5) Skewness: $\gamma_1^0 = \dfrac{\beta_2^*}{V_0^{3/2}}$ , where $\beta_2^* = \dfrac{\lambda}{\sqrt{\lambda^2+1}}\tanh(\pi\lambda)\sin\left(\tan^{-1}\left(\dfrac{1}{\lambda}\right) - 2\tan^{-1}\left(\dfrac{1}{2\lambda}\right)\right)$ .

(6) Kurtosis: $\gamma_2^* = \dfrac{\alpha_2^* - (1-V_0)^4}{V_0^{\,2}}$, where

$$\alpha_2^* = \frac{\lambda}{\sqrt{\lambda^2+1}} \tanh(\pi\lambda) \cos\left( \tan^{-1}\left(\frac{1}{\lambda}\right) - 2\tan^{-1}\left(\frac{1}{2\lambda}\right)\right).$$

**Table 1. Population characteristics for selected values of the parameter.**

| Characteristic | | Parameter | | | | |
|---|---|---|---|---|---|---|
| | | $\lambda$=0.15 | $\lambda$=0.25 | $\lambda$=0.85 | $\lambda$=1 | $\lambda$=1.7 |
| Trigonometric moments | $\alpha_1$ | 0.1880 | 0.3050 | 0.7501 | 0.8030 | 0.9204 |
| | $\alpha_2$ | 0.0220 | 0.0580 | 0.4194 | 0.5000 | 0.7429 |
| | $\beta_1$ | 0.6267 | 0.6099 | 0.4412 | 0.4015 | 0.2707 |
| | $\beta_2$ | 0.1467 | 0.2353 | 0.4935 | 0.5000 | 0.4370 |
| Central trigonometric moments | $\overline{\alpha_1}$ | 0.6543 | 0.6819 | 0.8702 | 0.8970 | 0.9594 |
| | $\overline{\alpha_2}$ | 0.0624 | 0.1529 | 0.6351 | 0.7000 | 0.8612 |
| | $\overline{\beta_1}$ | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| | $\overline{\beta_2}$ | -0.1346 | -0.1882 | -0.1269 | -0.1000 | -0.0348 |
| Mean direction | $\mu$ | 1.2793 | 1.1071 | 0.5317 | 0.4636 | 0.2861 |
| Resultant length | $\rho$ | 0.6543 | 0.6819 | 0.8702 | 0.8978 | 0.9594 |
| Circular variance | $v$ | 0.3457 | 0.3181 | 0.1298 | 0.1022 | 0.0406 |
| Circular Standard deviation | $\sigma$ | 0.9212 | 0.8750 | 0.5272 | 0.4644 | 0.2879 |
| Coefficient of skewness | $\xi_1$ | -0.6620 | -0.0494 | -2.7138 | -3.0596 | -4.2548 |
| Coefficient of kurtosis | $\xi_2$ | -1.0109 | -0.6260 | 3.6573 | 4.8197 | 8.4807 |

## 4. ESTIMATION

In this section, the classical maximum likelihood method is employed to estimate the unknown parameter in SCWE$(\lambda)$. Let $\theta_1, \theta_2, \theta_3, ..., \theta_n$ be a random sample drawn from SCWE$(\lambda)$. The likelihood function is given by $L = \prod_{i=1}^{n} g(\theta; \lambda)$, and the log-likelihood function is given by

$$l = n\log(2\lambda) - 2\lambda\sum_{i=1}^{n}(\theta_i) - n\log\left(1-e^{-2\pi\lambda}\right). \qquad (6)$$

Differentiating w.r.to $\lambda$, we get

$$\frac{\partial l}{\partial \lambda} = \frac{n}{2\lambda} - 2\sum_{i=1}^{n}(\theta_i) - \frac{2\pi n e^{-2\pi\lambda}}{1-e^{-2\pi\lambda}} = 0. \qquad (7)$$

Due to the presence of non-linear terms, we cannot get the MLE of the parameters in closed form. So, we can solve these normal equations using numerical methods.

**Table 2. Results of the simulation study.**

| $\lambda = 0.5$ | | | | |
|---|---|---|---|---|
| Sample size $(n)$ | MLE | Bias | MSE | MRE |
| 30 | 0.51460 | 0.10082 | 0.01648 | 0.20164 |
| 80 | 0.50631 | 0.06151 | 0.00592 | 0.12301 |
| 100 | 0.50669 | 0.05829 | 0.00521 | 0.11658 |
| 200 | 0.49974 | 0.03855 | 0.00233 | 0.07711 |
| 350 | 0.50126 | 0.03046 | 0.00145 | 0.06092 |
| $\lambda = 0.75$ | | | | |
| 30 | 0.77056 | 0.12169 | 0.00854 | 0.09699 |
| 80 | 0.75907 | 0.07274 | 0.00854 | 0.09699 |
| 100 | 0.75376 | 0.06595 | 0.00708 | 0.06565 |
| 200 | 0.75731 | 0.04924 | 0.00384 | 0.04909 |
| 350 | 0.74410 | 0.03682 | 0.00209 | 0.04909 |
| | | $\lambda = 1.5$ | | |
| Sample size $(n)$ | MLE | Bias | MSE | MRE |
| 30 | 1.54648 | 0.23324 | 0.09360 | 0.15549 |
| 80 | 1.51888 | 0.14214 | 0.03129 | 0.09476 |
| 100 | 1.51449 | 0.12484 | 0.02536 | 0.08322 |
| 200 | 1.50450 | 0.08480 | 0.01155 | 0.05654 |
| 350 | 1.50244 | 0.06398 | 0.00651 | 0.04266 |
| $\lambda = 2$ | | | | |
| Sample size $(n)$ | MLE | Bias | MSE | MRE |
| 30 | 2.08771 | 0.30390 | 0.16186 | 0.15195 |
| 80 | 2.02082 | 0.18781 | 0.05792 | 0.09391 |
| 100 | 2.03052 | 0.16196 | 0.04222 | 0.08098 |
| 200 | 2.01067 | 0.11459 | 0.02084 | 0.05729 |
| 350 | 2.00582 | 0.08517 | 0.01152 | 0.04259 |
| | | $\lambda = 4$ | | |
| Sample size $(n)$ | MLE | Bias | MSE | MRE |
| 30 | 4.10073 | 0.60109 | 0.58487 | 0.15027 |
| 80 | 4.03788 | 0.36681 | 0.21737 | 0.09170 |
| 100 | 4.02039 | 0.32659 | 0.17168 | 0.08165 |
| 200 | 4.00942 | 0.22245 | 0.07959 | 0.05561 |
| 350 | 4.00535 | 0.17159 | 0.04615 | 0.04290 |
| $\lambda = 7$ | | | | |
| Sample size $(n)$ | MLE | Bias | MSE | MRE |
| 30 | 7.24123 | 1.11610 | 2.23427 | 0.15944 |
| 80 | 7.08789 | 0.61342 | 0.59421 | 0.08763 |
| 100 | 7.06402 | 0.55625 | 0.49523 | 0.07946 |
| 200 | 7.03301 | 0.39718 | 0.25400 | 0.05674 |
| 350 | 7.02911 | 0.30068 | 0.14206 | 0.04295 |

As shown in Table 1, we performed simulations under conditions of both high and low skewness and lambda values. The results indicate that as the sample size increases, the bias, mean squared error (MSE), and mean relative error (MRE) decrease across all parameter settings. This trend suggests that the estimators are becoming more precise and accurate, indicating consistency and asymptotic unbiasedness.

## 5. REAL DATA APPLICATION

In this part, we present an application of the SCWE distribution using a real data set of 133 feldspar laths (FisherB2, Fig. 5). Semicircular wrapped exponential distribution (Table 3) is compared with the wrapped modified Lindley (WML) [13], wrapped Lindley (WL) [9], transmuted wrapped exponential (TWE) [10], and wrapped exponential (WE) [1]. Based on maximum log-likelihood, Akaike information criterion (AIC), Bayesian information criterion (BIC), consistent Akaike information criterion (CAIC), Hannan Information criterion (HQIC), and the Kolmogorov-Smirnov (K-S) statistic (given in Table 4) along with corresponding p-values, indicate that the SCWE distribution provides a more reliable and accurate fit to the data.

**Table 3. Estimates and standard errors (in parentheses) of the parameters for the dataset**

| Model | $\alpha$ | $\lambda$ |
|---|---|---|
| SCWE | - | 0.06074219(0.04798034) |
| WML | - | 0.7287109(0.05581266) |
| TWE | 0.8807770(0.07738507) | -0.6874853(0.12415055) |
| WL | - | 0.9878906(0.07126161) |
| WE | - | 0.06264834(0.06561712) |

**Table 4. Criteria and goodness-of-fit statistics for the dataset.**

| Model | -L | AIC | CAIC | BIC | HQIC | KS (p-value) |
|---|---|---|---|---|---|---|
| SCWE | 151.4490 | 304.8877 | 304.9183 | 304.7780 | 306.0623 | 0.0823(0.3286) |
| WML | 176.2034 | 354.4068 | 354.4374 | 360.2256 | 356.7940 | 0.1119(0.0713) |
| TWE | 175.2225 | 354.4449 | 354.5372 | 360.2256 | 356.7940 | 0.1119(0.0713) |
| WL | 176.6560 | 355.3121 | 355.3426 | 358.2024 | 356.4866 | 0.1368(0.0137) |
| WE | 182.1457 | 366.2914 | 366.3219 | 369.1816 | 367.4659 | 0.1749(0.0058) |

According to the KS values in Table 4, the goodness of fit of wrapped Lindley (WL) [9], wrapped modified Lindley (WML) [13], and transmuted wrapped exponential (TWE) [10] models cannot be rejected. However, compared to the other four distributions based on -L, AIC, CAIC, BIC, and HQIC values, the SCWE distribution fits better.

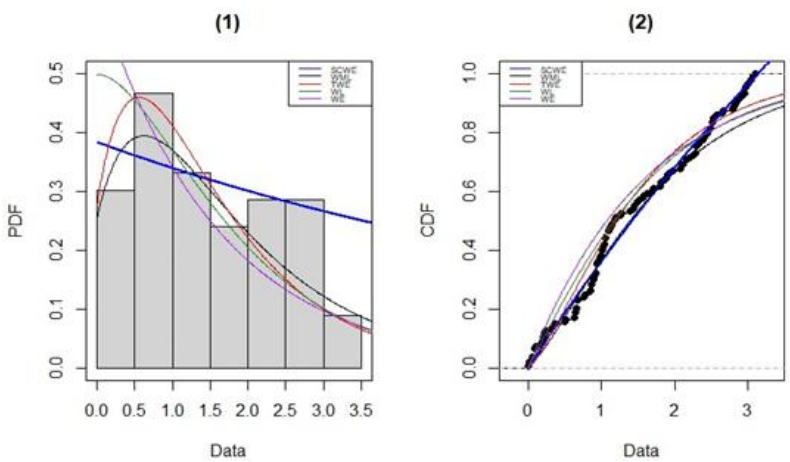


**Figure 5. Estimated pdfs and cdfs of the distributions for the dataset FisherB2.**

## 6. CONCLUSION

This study extends the wrapped exponential distribution to the axial domain, offering new insights into circular data analysis, especially for arcs of varying lengths. By deriving a closed-form characteristic function and examining key properties and estimation methods, we provide both theoretical and practical value. Our simulation results and real-data application demonstrate the effectiveness of the proposed model, with particular emphasis on the semicircular case, making it a useful tool for analyzing axial phenomena.

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