ORIGINAL PAPER

SOME PROPERTIES OF ANALYTIC FUNCTIONS ASSOCIATED WITH HURWITZ-ZETA FUNCTION

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Abstract. *The study of the geometric properties of analytic functions and their numerous applications across a variety of mathematical fields, including fractional calculus, probability distributions, and special functions, has drawn significant attention to Geometric Function Theory (GFT), one of the most prominent branches of complex analysis, in recent years. In this paper, we introduce a new subclass of uniformly convex functions with negative coefficients, defined in terms of the Hurwitz-Zeta function. We obtain the coefficient bounds, convex linear combinations, radii of close-to-convexity, starlikeness and integral means inequality for functions belonging to the class. Furthermore, we obtain neighbourhood results for this class.*

Keywords: *analytic; starlike; convex; convolution; neighbourhood.*
Mathematics Subject Classification: *30C45.*

1. INTRODUCTION

Let A denote the class of analytic functions u defined on the unit disk $U = \{z: |z| < 1\}$ with normalization $u(0) = 0$ and $u'(0) = 1$. Such a function has the Taylor series expansion about the origin in the form

$$u(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

denoted by S, the subclass of A consisting of functions that are univalent in U.

A function $u(z) \in A$ is known as starlike and convex of order ϑ if its satisfies

$$\Re\left\{\frac{zu'(z)}{u(z)}\right\} > \vartheta, (z \in U),$$

and

$$\Re\left\{1 + \frac{zu''(z)}{u'(z)}\right\} > \vartheta, (z \in U),$$

for specific $\vartheta (0 \leq \vartheta < 1)$ respectively, and we express by $S^*(\vartheta)$ and $K(\vartheta)$ the subclass of A

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Also, indicate by T the subclass of A made up of functions of this form

$$u(z) = z - \sum_{n=2}^{\infty} a_n z^n,\ (a_n \geq 0,\ z \in U) \tag{1.2}$$

and let $T^*(\vartheta) = T \cap S^*(\vartheta), C(\vartheta) = T \cap K(\vartheta)$. There are interesting properties in the $T^*(\vartheta)$ and $C(\vartheta)$ classes and were thoroughly studied by Silverman [1] and others [2].

For $u \in A$ given by (1.1) and $g(z)$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n$$

their convolution (or Hadamard product), denoted by $(u * g)$, is defined as

$$(u * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * u)(z), (z \in U)$$

note that $u * g \in A$.

A function $u \in A$ is said to be in $k - US(\gamma)$, the class of $k -$uniformly starlike functions of order $\gamma, 0 \leq \gamma < 1$, if satisfies the condition

$$\Re\left\{\frac{zu'(z)}{u(z)}\right\} > k\left|\frac{zu'(z)}{u(z)} - 1\right| + \gamma, (k \geq 0)$$

and a function $u \in A$ is said to be in $k - UC(\gamma)$, the class of $k -$uniformly convex functions of order $\gamma, 0 \leq \gamma < 1$, if satisfies the condition

$$\Re\left\{1 + \frac{zu''(z)}{u'(z)}\right\} > k\left|\frac{zu''(z)}{u'(z)}\right| + \gamma, (k \geq 0).$$

Uniformly starlike and uniformly convex functions were first introduced by Goodman [3-4] and then studied by various authors [5-9]. In [10], Mustafa and Darus recently introduced a new generalized integral operator $\mathfrak{J}^{\alpha}_{\rho,b}u(z)$, as shown below.

Definition 1.1. [11] A general Hurwitz- Lerch zeta function $\Phi(z, \rho, b)$ defined by

$$\Phi(z, \rho, b) = \sum_{n=0}^{\infty} \frac{z^n}{(n + b)^{\rho}},$$

where $(\rho \in C, b \in C - \mathbb{Z}_{0}^{-})$ when $|z| < 1$, and $\Re(\mathrm{b}) > 1$ when $(|z| = 1)$. We define the function:

$$\Phi^*(z, \rho, b) = \left(b^{\rho} z \Phi(z, \rho, b)\right) * u(z),$$

then

$$\Phi^*(z,\rho,b) = z + \sum_{n=2}^{\infty} \frac{a_n}{(n+b-1)^{\rho}} z^n$$

Definition 1.2. [12] Let the function u be analytic in a simply connected domain of the z-plane containing the origin. The fractional derivative of u of order α is defined by

$$D_z^{\alpha} u(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{u(t)}{(z-t)^{\alpha}} dt, (0 \le \alpha < 1),$$

where the multiplicity of $(z-t)^{-\alpha}$ is removed by requiring $log(z-t)$ to be real when $(z-t) > 0$.

Using Definition 1.2 and its known extensions involving fractional derivatives and fractional integrals, Owa and Srivastava [13] introduced the operator $\Omega^{\alpha}: A \to A$ which is known as an extension of the fractional derivative and the fractional integral, as follows:

$$\begin{aligned} \Omega^{\alpha} u(z) &= \Gamma(2-\alpha) z^{\alpha} D_z^{\alpha} u(z), (\alpha \ne 2,3,4,\dots) \\ &= z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)} a_n z^n, (z \in U) \end{aligned}$$

For $\alpha \in \mathbb{C}, b \in \mathbb{C} - \mathbb{Z}_{\mathbb{O}}^{-}$, and $0 \le \alpha < 1$, the generalized integral operator $\mathfrak{I}_{\rho,\mathfrak{b}}^{\alpha} u: A \to A$, is defined by

$$\begin{aligned} \mathfrak{I}_{\rho,\mathfrak{b}}^{\alpha} u(z) &= \Gamma(2-\alpha) z^{\alpha} D_z^{\alpha} \Phi^*(z,\alpha,b), (\alpha \ne 2,3,4,\dots) \\ &= z + \sum_{n=2}^{\infty} \Theta\,(n,\rho,b,\alpha)\; a_n z^n, (z \in U). (1.3) \end{aligned}$$

where $\Theta(\mathrm{n}) = \Theta(n,\rho,b,\alpha) = \frac{\Gamma(n+1) z^{\alpha} D_z^{\alpha} \Phi^*(z,\alpha,b)}{\Gamma(n+1-\alpha)} \left(\frac{b}{n-1+b}\right)^{\rho}$. Note that: $\mathfrak{I}_{0,\mathfrak{b}}^{0} u(z) = u(z)$.

Special cases of this operator include:

(i) $\mathfrak{I}_{0,\mathfrak{b}}^{\alpha} u(z) \equiv \Omega^{\alpha} u(z)$ is the Owa and Srivastava operator [13].
(ii) $\mathfrak{I}_{\rho,\mathfrak{b}+1}^{0} u(z) \equiv J_{\rho,b} u(z)$ is the Srivastava and Attiya integral operator [14].
(iii) $\mathfrak{I}_{1,1}^{0} u(z) \equiv A(u)(z)$ is the Alexander integral operator [15].
(iv) $\mathfrak{I}_{\rho+1,1}^{0} u(z) \equiv L(u)(z)$ is the Libera integral operator [16].
(v) $\mathfrak{I}_{1,\delta}^{0} u(z) \equiv L_{\delta}(u)(z)$ is the Bernardi integral operator [17].
(vi) $\mathfrak{I}_{\sigma,2}^{0} u(z) \equiv I^{\sigma} u(z)$ is the Jung-Kim-Kim-Srivastava integral operator [18].

Now, by making use of the Hurwitz - Lerch zeta operator $\mathfrak{I}_{\rho,b}^{\alpha} u$, we define a new subclass of functions belonging to the class A.

Definition 1.3. The function $u(z)$ of the form (1.1) is in the class $\Omega_{\rho.\mathrm{b}}^{\alpha}(\mu,\gamma,\ell)$, if it satisfies the inequality

$$\Re\left\{\frac{z\left(\mathfrak{I}_{\rho,b}^{\alpha} u(z)\right)'}{(1-\mu)z + \mu \mathfrak{I}_{\rho,b}^{\alpha} u(z)} - \gamma\right\} > \ell \left|\frac{z\left(\mathfrak{I}_{\rho,b}^{\alpha} u(z)\right)'}{(1-\mu)z + \mu \mathfrak{I}_{\rho,b}^{\alpha} u(z)} - 1\right|$$

for $0 \leq \mu \leq 1$, $0 \leq \gamma \leq 1$and $\ell \geq 0$.

Further we define

$$T\Omega^{\alpha}_{\rho.b}(\mu,\gamma,\ell) = \Omega^{\alpha}_{\rho.b}(\mu,\gamma,\ell) \cap T$$

The aim of the present paper is to study the coefficient bounds, radii of close-to-convex and starlikeness convex linear combinations and integral means inequalities of the $T\Omega^{\alpha}_{\rho.b}(\mu,\gamma,\ell)$.

2. COEFFICIENT BOUNDS

Theorem 2.1. Let the function $u(z) \in \Omega^{\alpha}_{\rho.b}(\mu,\gamma,\ell)$. Then

$$\sum_{n=2}^{\infty}[n(1+\ell) - \mu(\gamma+\ell)]\,\Theta(n)|a_n| \leq 1-\gamma \tag{2.1}$$

where $0 \leq \mu \leq 1$, $0 \leq \gamma \leq 1$, $\ell \geq 0$ and $\Theta(n)$ is given by (1.3).

Proof: It suffices to show that

$$\ell\left|\frac{z\left(\mathfrak{J}^{\alpha}_{\rho,b}u(z)\right)'}{(1-\mu)z+\mu\mathfrak{J}^{\alpha}_{\rho,b}u(z)} - 1\right| - \Re\left\{\frac{z\left(\mathfrak{J}^{\alpha}_{\rho,b}u(z)\right)'}{(1-\mu)z+\mu\mathfrak{J}^{\alpha}_{\rho,b}u(z)} - 1\right\} \leq 1-\gamma.$$

We have

$$\begin{aligned}
&\ell\left|\frac{z\left(\mathfrak{J}^{\alpha}_{\rho,b}u(z)\right)'}{(1-\mu)z+\mu\mathfrak{J}^{\alpha}_{\rho,b}u(z)} - 1\right| - \Re\left\{\frac{z\left(\mathfrak{J}^{\alpha}_{\rho,b}u(z)\right)'}{(1-\mu)z+\mu\mathfrak{J}^{\alpha}_{\rho,b}u(z)} - 1\right\} \\
&\leq (1+\ell)\left|\frac{z\left(\mathfrak{J}^{\alpha}_{\rho,b}u(z)\right)'}{(1-\mu)z+\mu\mathfrak{J}^{\alpha}_{\rho,b}u(z)} - 1\right| \\
&\leq \frac{(1+\ell)\sum_{n=2}^{\infty}(n-\mu)\,\Theta(n)|a_n||z|^{n-1}}{1-\sum_{n=2}^{\infty}\mu\,\Theta(n)|a_n||z|^{n-1}} \\
&\leq \frac{(1+\ell)\sum_{n=2}^{\infty}(n-\mu)\,\Theta(n)|a_n|}{1-\sum_{n=2}^{\infty}\mu\,\Theta(n)|a_n|}.
\end{aligned}$$

The last expression is bounded above by $(1-\gamma)$, if

$$\sum_{n=2}^{\infty}[n(1+\ell) - \mu(\gamma+\ell)]\,\Theta(n)|a_n| \leq 1-\gamma$$

and the proof is complete.

Theorem 2.2. Let $0 \leq \mu \leq 1$, $0 \leq \gamma \leq 1$ and $\ell \geq 0$. Then a function $u \in T\Omega_{\rho.b}^{\alpha}(\mu,\gamma,\ell) \Leftrightarrow$

$$\sum_{n=2}^{\infty}[n(1+\ell)-\mu(\gamma+\ell)]\,\Theta(n,\wp,\varsigma)|a_n| \leq 1-\gamma \tag{2.2}$$

where $\Theta(n)$ is given by (1.3) .

Proof: In view of Theorem 2.1, we need only to prove the necssity. If $u \in T\Omega_{\rho.b}^{\alpha}(\mu,\gamma,\ell)$ and z is real, then

$$\Re\left\{\frac{1-\sum_{n=2}^{\infty} n\,\Theta(n)a_n z^{n-1}}{1-\sum_{n=2}^{\infty} \mu\,\Theta(n)a_n z^{n-1}}-\gamma\right\} > \ell\left|\frac{\sum_{n=2}^{\infty}(n-\mu)\,\Theta(n)a_n z^{n-1}}{1-\sum_{n=2}^{\infty} \mu\,\Theta(n)a_n z^{n-1}}\right|.$$

Letting $z \to 1$ along the real axis, we obtain the desired inequality

$$\sum_{n=2}^{\infty}[n(1+\ell)-\mu(\gamma+1)]\,\Theta(n)|a_n| \leq 1-\gamma.$$

Corollary 2.1. If $u(z) \in T\Omega_{\rho.b}^{\alpha}(\mu,\gamma,\ell)$ then

$$|a_n| \leq \frac{1-\gamma}{[n(1+\ell)-\mu(\gamma+\ell)]\Theta(n)} \tag{2.3}$$

where $0 \leq \mu \leq 1$, $0 \leq \gamma \leq 1$, $\ell \geq 0$.

Equality holds for the function

$$u(z) = z - \frac{1-\gamma}{[n(1+\ell)-\mu(\gamma+\ell)]\Theta(n)}z^n. \tag{2.4}$$

Theorem 2.3. Let $u_1(z) = z$ and $u_n(z) = z - \frac{1-\gamma}{[n(1+\ell)-\mu(\gamma+\ell)]\Theta(n)}z^n, n \geq 2$. Then $u(z) \in T\Omega_{\rho.b}^{\alpha}(\mu,\gamma,\ell)$ if and only if it can be expressed in the form

$$u(z) = \sum_{n=1}^{\infty} w_n\,u_n(z), w_n \geq 0, \sum_{n=1}^{\infty} w_n = 1. \tag{2.5}$$

Proof: Suppose $u(z)$ can be written as in (2.5) Then

$$u(z) = z - \sum_{n=2}^{\infty} w_n \frac{1-\gamma}{[n(1+\ell)-\mu(\gamma+\ell)]\Theta(n)}z^n.$$

Now,

$$\sum_{n=2}^{\infty} w_n \frac{(1-\gamma)[n(1+\ell)-\mu(\gamma+\ell)]\Theta(n)}{(1-\gamma)[n(1+\ell)-\mu(\gamma+\ell)]\Theta(n)}$$
$$= \sum_{n=2}^{\infty} w_n = 1 - w_1 \leq 1.$$

thus $u(z) \in T\Omega_{\rho.b}^{\alpha}(\mu,\gamma,\ell)$.

Conversely, let $u(z) \in T\Omega_{\wp}^{\varsigma}(\mu,\gamma,\ell)$. Then by using (2.3), we get

$$w_n = \frac{[n(1+\ell)-\mu(\gamma+\ell)]\Theta(n)}{(1-\gamma)} a_n, n \geq 2$$

and $w_1 = 1 - \sum_{n=2}^{\infty} w_n$. Then we have $u(z) = \sum_{n=1}^{\infty} w_n\, u_n(z)$ and hence this completes the proof of theorem.

Theorem 2.4. The class $T\Omega_{\rho.b}^{\alpha}(\mu,\gamma,\ell)$ is a convex set.

Proof: Let the function $u_j(z) = z - \sum_{n=2}^{\infty} a_{n,j}\, z^n, a_{n,j} \geq 0, j = 1,2$ be in the class $T\Omega_{\rho.b}^{\alpha}(\mu,\gamma,\ell)$It is sufficient to show that the function $h(z)$ defined by

$$h(z) = \xi u_1(z) + (1-\xi)u_2(z), 0 \leq \xi < 1,$$

in the class $T\Omega_{\rho.b}^{\alpha}(\mu,\gamma,\ell)$. Since

$$h(z) = z - \sum_{n=2}^{\infty} \left[\xi a_{n,1} + (1-\xi)a_{n,2}\right] z^n.$$

An easy computation with the aid of Theorem 2.2, gives

$$\sum_{n=2}^{\infty}[n(1+\ell)-\mu(\gamma+\ell)] \quad \xi\Theta(n)a_{n,1} + \sum_{n=2}^{\infty}[n(1+\ell)-\mu(\gamma+\ell)]\,(1-\xi)\Theta(n)a_{n,2}$$
$$\leq \xi(1-\gamma) + (1-\xi)(1-\gamma)$$
$$\leq (1-\gamma)$$

which implies that $h \in T\Omega_{\rho.b}^{\alpha}(\mu,\gamma,\ell)$.

Hence $T\Omega_{\wp}^{\varsigma}(\mu,\gamma,\ell)$ is convex.

3. RADII OF CLOSE-TO-CONVEXITY AND STARLIKENESS

In this section, we obtain the radii of close-to-convexity and starlikeness for the class $T\Omega_{\rho.b}^{\alpha}(\mu,\gamma,\ell)$.

Theorem 3.1. Let the function $u(z) \in T\Omega^{\alpha}_{\rho.b}(\mu,\gamma,\ell)$. Then $u(z)$ is close to convex of order $\delta(0 \le \delta < 1)$ in the disc $|z| < r_1$, where $r_1 = \inf_{n\ge 2}\left[\frac{(1-\delta)\sum_{n=2}^{\infty}[n(1+\ell)-\mu(\gamma+\ell)]\Theta(n)}{n(1-\gamma)}\right]^{1/n-1}, n \ge 2.$ The result is sharp with the external function $u(z)$ *is* given by (2.4).

Proof: Given $u \in T$ and u is close to convex of order δ, we have

$$|u'(z) - 1| < 1 - \delta. \tag{3.1}$$

For the left-hand side of (3.1), we have

$$|u'(z) - 1| \le \sum_{n=2}^{\infty} n\, a_n |z|^{n-1}.$$

The last expression is less than $1 - \delta$

$$\sum_{n=2}^{\infty} \frac{n}{1-\delta} a_n |z|^{n-1} \le 1.$$

Using the fact that $u(z) \in T\Omega^{\alpha}_{\rho.b}(\mu,\gamma,\ell)$ if and only if

$$\sum_{n=2}^{\infty} \frac{[n(1+\ell) - \mu(\gamma+\ell)]\Theta(n)}{1-\gamma} a_n \le 1.$$

We can see that (3.1) is true if

$$\frac{n}{1-\delta}|z|^{n-1} \le \frac{[n(1+\ell) - \mu(\gamma+\ell)]\Theta(n)}{1-\gamma}$$

or, equivalently

$$|z| \le \left\{\frac{(1-\delta)[n(1+\ell) - \mu(\gamma+\ell)]\Theta(n)}{n(1-\gamma)}\right\}^{1/n-1}$$

which completes the proof.

Theorem 3.2. Let the function $u(z) \in T\Omega^{\alpha}_{\rho.b}(\mu,\gamma,\ell)$. Then $u(z)$ is starlike of order $\delta(0 \le \delta < 1)$ *in the disc* $|z| < r_2$, where $r_2 = \inf_{n\ge 2}\left[\frac{(1-\delta)\sum_{n=2}^{\infty}[n(1+\ell)-\mu(\gamma+\ell)]\Theta(n)}{(n-\delta)(1-\gamma)}\right]^{1/n-1}.$

The result is sharp with the extremal function $u(z)$ is given by (2.4)

Proof: Given $u \in T$ and u is starlike of order δ, we have

$$\left|\frac{zu'(z)}{u(z)}-1\right|<1-\delta. \tag{3.2}$$

For the left-hand side of (3.2), we have

$$\left|\frac{zu'(z)}{u(z)}-1\right|\leq\sum_{n=2}^{\infty}\frac{(n-1)a_n|z|^{n-1}}{1-\sum_{n=2}^{\infty}a_n\,|z|^{n-1}}.$$

The last expression is less than $1-\delta$ if

$$\sum_{n=2}^{\infty}\frac{n-\delta}{1-\delta}a_n|z|^{n-1}<1.$$

Using the fact that $u(z)\in T\Omega_{\mathrm{p.b}}^{\alpha}(\mu,\gamma,\ell)$ if and only if

$$\sum_{n=2}^{\infty}\frac{[n(1+\ell)-\mu(\gamma+\ell)]\Theta(n)}{1-\gamma}a_n\leq 1.$$

We can say (3.2) is true, if

$$\sum_{n=2}^{\infty}\frac{n-\delta}{1-\delta}|z|^{n-1}\leq\frac{[n(1+\ell)-\mu(\gamma+\ell)]\Theta(n)}{1-\gamma}$$

or equivalently

$$|z|^{n-1}\leq\frac{(1-\delta)[n(1+\ell)-\mu(\gamma+\ell)]\Theta(n)}{(n-\delta)(1-\gamma)}$$

which yields the family's starlikeness.

4. INTEGRAL MEANS INEQUALITIES

In [1], Silverman found that the function $u_2(z)=z-\frac{z^2}{2}$ is often extremal over the family T. He applied this function to resolve his integral means inequality conjectured [19] and settled in [20], that

$$\int_0^{2\pi}\left|u\left(re^{i\varphi}\right)\right|^{\tau}d\varphi\leq\int_0^{2\pi}\left|u_2\left(re^{i\varphi}\right)^{n}\right|^{\tau}d\varphi$$

for all $u\in T$, $\tau>0$ and $0<r<1$. In [20], he also proved his conjuncture for the subclasses $T^*(\alpha)$ and $C(\alpha)$ of T.

Now, we prove Silverman's conjecture for the class of functions $T\Omega^{\alpha}_{\rho.b}(\mu,\gamma,\ell)$. We need the concept of subordination between analytic functions and Littlewood's subordination theorem [21].

Two functions u and v, which are analytic in U, the function u is said to be subordinate to v in U, if there exists a function w analytic in U with $w(0)=0, |w(z)|<1, (z\in U)$ such that $u(z)=v\big(w(z)\big), (z\in U)$. We denote this subordination by $u(z)\prec v(z)$, ($\prec$ denote subordination).

Lemma 4.1. If the function u *and* v are analytic in U with $u(z)\prec v(z)$, then for $\tau>0$ and $z=re^{i\varphi}$, $0<r<1$ $\int_0^{2\pi}\left|v\big(re^{i\varphi}\big)\right|^{\tau}d\varphi\le\int_0^{2\pi}\left|u\big(re^{i\varphi}\big)\right|^{\tau}d\varphi$. Now, we discuss the integral means inequalities for functions u *in* $T\Omega^{\alpha}_{\rho.b}(\mu,\gamma,\ell)$

Theorem 4.1. $u\in T\Omega^{\alpha}_{\rho.b}(\mu,\gamma,\ell)$, $0\le\mu<1$, $0\le\gamma<1$ *and* $u_2(z)$ be defined by

$$u_2(z)=z-\frac{1-\gamma}{\phi(2)}z^2 \tag{4.1}$$

Proof: For $u(z)=z-\sum_{n=2}^{\infty}a_n\,z^n$, (4.1) is equivalent to

$$\int_0^{2\pi}\left|1-\sum_{n=2}^{\infty}a_n\,z^{n-1}\right|^{\tau}d\varphi\le\int_0^{2\pi}\left|1-\frac{1-\gamma}{\phi(2)}z\right|^{\tau}d\varphi.$$

By Lemma (4.1), it is enough to prove that

$$1-\sum_{n=2}^{\infty}a_n\,z^{n-1}\prec 1-\frac{1-\gamma}{\phi(2)}z.$$

Assuming

$$1-\sum_{n=2}^{\infty}a_n\,z^{n-1}=1-\frac{1-\gamma}{\phi(2)}w(z)$$

and using (2.2), we obtain

$$|w(z)|=\left|\sum_{n=2}^{\infty}\frac{\phi(n)}{1-\gamma}a_nz^{n-1}\right|\le|z|\sum_{n=2}^{\infty}\frac{\phi(n)}{1-\gamma}a_n\le|z|<1,$$

where

$$\phi(n)=[n(1+\ell)-\mu(\gamma+1)\Theta(n).$$

This completes the proof.

5. NEIGHBOURHOOD PROPERTY

Following the earlier investigations by Goodman [22], Kazimogulu [23] and Ruscheweyh [24], we define the (n,δ)- neighbourhood of a function $u(z) \in T$ by

$$N_\delta(u) = \left\{ g \in T : g(z) = z - \sum_{n=2}^{\infty} b_n z^n \text{ and } \sum_{n=2}^{\infty} n\,|a_n - b_n| \le \delta \right\}. \tag{5.1}$$

In particular, if $e(z) = z$, we have

$$N_\delta(e) = \left\{ g \in T : g(z) = z - \sum_{n=2}^{\infty} b_n z^n \text{ and } \sum_{n=2}^{\infty} n\,|b_n| \le \delta \right\}.$$

Now we determine the neighbourhood for each of the classes $T\Omega_{\wp}^{\varsigma}(\mu,\gamma,\ell)$ which we define as follows. A function $u \in T$ is said to be in the class $T\Omega_{\wp}^{\varsigma}(\mu,\gamma,\ell,\xi)$ if there exists a function $g \in T\Omega_{\wp}^{\varsigma}(\mu,\gamma,\ell)$ such that

$$\left| \frac{u(z)}{g(z)} - 1 \right| \le 1 - \xi, \quad (z \in U, 0 \le \xi < 1).$$

Theorem 5.1. If $g \in T\Omega_{\wp}^{\varsigma}(\mu,\gamma,\ell)$ and

$$\xi = 1 - \frac{\delta[2(1+\ell) - \mu(\gamma+1)]\Theta(2)}{2\left[\left(2(1+\ell) - \mu(\gamma+\ell)\right) - (1-\gamma)\right]} \tag{5.2}$$

then $N_\delta(g) \subset T\Omega_{\wp}^{\varsigma}(\mu,\gamma,\ell,\xi)$.

Proof: Suppose $u(z) \in N_\delta(g)$. We find from (5.1) that

$$\sum_{n=2}^{\infty} n\,|a_n - b_n| \le \delta$$

which implies that

$$\sum_{n=2}^{\infty} |a_n - b_n| \le \frac{\delta}{2}, \quad (n \in \mathbb{N}).$$

Next, since $g(z) \in T\Omega_{\wp}^{\varsigma}(\mu,\gamma,\ell)$, we have

$$\sum_{n=2}^{\infty} |b_n| \le \frac{1-\gamma}{[2(1+\ell) - \mu(\gamma+\ell)]\Theta(2)}$$

so that

$$\begin{aligned}\left|\frac{u(z)}{g(z)}-1\right| &< \frac{\sum_{n=2}^{\infty}|a_n-b_n|}{1-\sum_{n=2}^{\infty}|b_n|} \\ &\leq \frac{\delta}{2}\left[\frac{[2(1+\ell)-\mu(\gamma+\ell)]\Theta(2)}{[2(1+\ell)-\mu(\gamma+\ell)]\Theta(2)-(1-\gamma)}\right] \\ &\leq 1-\xi,\end{aligned}$$

provided that ξ is given by (5.2). Thus, $u(z) \in T\Omega_{\wp}^{\varsigma}(\mu,\gamma,\ell,\xi)$ for ξ given by (5.2). This completes the proof.

6. CONCLUSIONS

In this study, we explored a particular subclass of analytic functions closely tied to the Hurwitz-Zeta function. Our primary focus was to investigate the properties, behaviors, and potential applications of these functions in various mathematical contexts. We demonstrated several key properties of this subclass, including its structural characteristics and functional equations. We provided explicit representations and derived important identities that underscore the relationship between these analytic functions and the Hurwitz-Zeta function. These findings contribute to a deeper understanding of the underlying mathematical structures and offer new perspectives on their potential applications in number theory and complex analysis.

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