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ORIGINAL PAPER

NEW ASYMPTOTIC EXPANSIONS AND INEQUALITIES FOR THE LANDAU CONSTANTS

SHUN WEI XU[1], CHAO PING CHEN[1], CRISTINEL MORTICI[2,3,4,*]

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Abstract. *The Landau constants are defined by*

$$G_n=\sum_{k=0}^{n}\frac{1}{16^k}\binom{2k}{k}^2,\ \ n\in\mathbb{N}_0:=\mathbb{N}\cup\{0\};\ \mathbb{N}:=\{1,2,3,\dots\}.$$

These constants play an important role in complex analysis. In this paper, we establish new asymptotic expansions and inequalities for the Landau constants.

Keywords: *Landau constants; asymptotic expansion; inequality; Psi function; Bernoulli polynomials.*

Mathematics Subject Classification: *41A60; 26D15.*

1. INTRODUCTION AND MOTIVATION

Throughout this paper, $\mathbb{N}$ represents the set of positive integers and $\mathbb{N}_0:=\mathbb{N}\cup\{0\}$. The Landau constants are defined by

$$G_n=\sum_{k=0}^{n}\frac{1}{16^k}\binom{2k}{k}^2,\quad n\in\mathbb{N}_0,$$

which play an important role in the theory of complex analysis. Landau himself [1] studied the asymptotic behavior of G_n and showed that

$$G_n\sim\frac{1}{\pi}\ln n,\quad n\to\infty.$$

Since then, the problem of approximation of the Landau constants has attracted many researchers. It is found that the investigation for the approximation of G_n has two main directions: One is to find sharper bounds of G_n, for all $n\in\mathbb{N}$. The other is to obtain asymptotic formulas of G_n for large $n\in\mathbb{N}$.

Watson [2] proved the following asymptotic formula:

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[1] Henan Polytechnic University, School of Mathematics and Information Science, 454003 Henan, China. Email: xswmath@163.com; chenchaoping@sohu.com.

[2] Valahia University of Targoviste, Faculty of Sciences and Arts, 130004 Targoviste, Romania.

[3] National University of Science and Technology Politehnica of Bucharest, Doctoral School of Applied Sciences, 060042 Bucharest, Romania.

[4] Academy of Romanian Scientists, 050085 Bucharest, Romania.

[*] *Corresponding author: cristinel.mortici@valahia.ro, cristinel.mortici@hotmail.com.*

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$$G_n = \frac{1}{\pi}\ln(n+1) + c_0 - \frac{1}{4\pi(n+1)} + O\left(\frac{1}{n^2}\right), \quad n \to \infty \tag{1.1}$$

with

$$c_0 = \frac{1}{\pi}(\gamma + 4\ln 2) = 1.0662758532089143543451101966157\ldots, \tag{1.2}$$

where γ denotes the Euler-Mascheroni constant (see, e.g., [3, Section 1.2]). In what follows, c_0 is given in (1.2).

Diverse sharp inequalities and asymptotic expansions for G_n have been established (see, e.g., [4-28]). For example, Zhao [27, Theorem 1] established the following two-sided inequality for all $n \in \mathbb{N}$:

$$\begin{aligned} &\frac{1}{\pi}\ln(n+1) + c_0 - \frac{1}{4\pi(n+1)} + \frac{5}{192\pi(n+1)^2} < G_n \\ &< \frac{1}{\pi}\ln(n+1) + c_0 - \frac{1}{4\pi(n+1)} + \frac{5}{192\pi(n+1)^2} + \frac{3}{128(n+1)^3}, \end{aligned} \tag{1.3}$$

which implies Watson's asymptotic formula (1.1) [2]. Mortici [21, Theorem 2] improved Zhao's result (1.3) [27].

Nemes and Nemes [23] derived full asymptotic expansions of G_n using a formula in [15]. They also conjectured a symmetry property of the coefficients in the expansion. Subsequently, Nemes himself [22] proved this conjecture. Nemes [22, Theorem 1.1] proved that, for $0 < h < 3/2$, the Landau constants G_n have the following asymptotic expansion:

$$G_n \sim \frac{1}{\pi}\ln(n+h) + c_0 - \sum_{k=1}^{\infty} \frac{g_k(h)}{(n+h)^k}, \quad n \to \infty, \tag{1.4}$$

where the coefficients $g_k(h)$ are given by

$$g_k(h) = \frac{1}{\pi k}\sum_{j=0}^{k}\left(\binom{k}{j} B_{k-j}\left(h - \frac{1}{2}\right)\sum_{m=0}^{j}(-1)^{j+m}\binom{2m}{m}^2 \frac{m!\,S(j,m)}{16^m}\right). \tag{1.5}$$

Here, $S(j,m)$ denotes Stirling numbers of the second kind defined by the generating function [3, p.78]:

$$\frac{(e^x - 1)^m}{m!} = \sum_{k=m}^{\infty} S(k,m)\frac{x^k}{k!},$$

and the $B_k(t)$ denotes Bernoulli polynomials defined by the generating function [3, p. 81]:

$$\frac{ze^{tz}}{e^z - 1} = \sum_{n=0}^{\infty} B_n(t)\frac{z^{\mathrm{n}}}{n!}, \quad |z| < 2\pi.$$

The rational numbers $B_n = B_n(0)$ are called Bernoulli numbers.

In fact, (1.4) also holds for $h = 0$. Nemes [22] showed that for $0 < h < 3/2$,

$$g_k(h) = (-1)^k g_k\left(\frac{3}{2} - h\right), \quad \text{for } k \in \mathbb{N},$$

which implies

$$g_{2k-1}\left(\frac{3}{4}\right) = 0, \quad \text{for } k \in \mathbb{N}. \tag{1.6}$$

Moreover, the case $h = \frac{3}{4}$ is investigated in detail in Nemes' paper [22].

Li et al. [19] gave the following recursive formula to determine the coefficients $G_k(h)$ in (1.4):

$$\begin{aligned}(s-2)^2 g_{s-2}(h) = (-1)^s \left\{\frac{2^s-2}{s} - \frac{1}{4}\sum_{\ell=1}^{s-1} \frac{\ell+7-4h}{s-\ell}(2-h)^{\ell-2}\right\} \\ + \sum_{k=3}^{s-1}\left\{(2-2^k)\binom{k-s}{k} + \sum_{\ell=1}^{k-1}\frac{\ell+7-4h}{4}(h-2)^{\ell-2}\binom{k-s}{k-l}\right\} g_{s-k}(h)\,, s \geq 3.\end{aligned} \tag{1.7}$$

In [19, Theorem 1], the coefficients of the expansion (1.4) are given iteratively in an explicit manner, and the conjecture of [23] is also confirmed. It is worth mentioning that Li et al. [19,20] considered an important special case and proved that

$$(-1)^{l+1}\left(G_n - \frac{1}{\pi}\ln\left(n+\frac{3}{4}\right) - c_0 - \frac{1}{\pi}\sum_{s=1}^{l-1}\frac{\beta_{2s}}{\left(n+\frac{3}{4}\right)^{2s}}\right) > 0 \tag{1.8}$$

for $l \in \mathbb{N}$ and $n \in \mathbb{N}_0$, where $(-1)^{s+1}\beta_{2s}$ are positive rational numbers, given explicitly in an iterative manner [20, Lemma 1]. The inequality (1.8) can be written as

$$\frac{1}{\pi}\ln\left(n+\frac{3}{4}\right) + c_0 + \frac{1}{\pi}\sum_{s=1}^{2m}\frac{\beta_{2s}}{\left(n+\frac{3}{4}\right)^{2s}} < G_n < \frac{1}{\pi}\ln\left(n+\frac{3}{4}\right) + c_0 + \frac{1}{\pi}\sum_{s=1}^{2m-1}\frac{\beta_{2s}}{\left(n+\frac{3}{4}\right)^{2s}} \tag{1.9}$$

for all $n \in \mathbb{N}_0$ and $m \in \mathbb{N}$. The first few coefficients β_{2s} are

$$\beta_2 = \frac{11}{192}, \quad \beta_4 = -\frac{1541}{122880}, \quad \beta_6 = \frac{63433}{8257536}, \quad \beta_8 = -\frac{9199901}{1006632960},$$
$$\beta_{10} = \frac{317959723}{17716740096}, \quad \beta_{12} = -\frac{14849190321163}{281406257233920}, \quad \beta_{14} = \frac{717209117969}{3298534883328}.$$

Inequality (1.9) implies that

$$G_n \sim \frac{1}{\pi}\ln\left(n+\frac{3}{4}\right) + c_0 + \frac{1}{\pi}\sum_{s=1}^{\infty}\frac{\beta_{2s}}{\left(n+\frac{3}{4}\right)^{2s}}\,, \quad n \to \infty. \tag{1.10}$$

Noting that (1.6) holds, we obtain from (1.4) and (1.10) that

$$-\pi g_{2l}\left(\frac{3}{4}\right) = \beta_{2l}\,, \quad \ell \geq 1. \tag{1.11}$$

Using the computer program MAPLE 11, we find, as $n \to \infty$,

$$G_n \sim \frac{1}{\pi}\ln\left(n+\frac{3}{4}\right) + c_0 \tag{1.12}$$

$$+\frac{1}{\pi}\left\{\frac{\frac{11}{192}}{n^2+\frac{3}{2}n+\frac{5501}{7040}}+\frac{\frac{89684299}{18166579200}}{\left\{n^2+\frac{3}{2}n+\frac{719149947443}{631377464960}\right\}^3}\right.$$
$$\left.+\frac{\frac{306469422974562781964 87}{38451704669013855436800 00}}{\left\{n^2+\frac{3}{2}n+\frac{14513829897086081062862741419735820039}{88041538751272211487811896911059616 00}\right\}^5}+\cdots\right\}.$$

From a computational viewpoint, the formula (1.12) is better than the formula (1.10). The first aim of the present paper is to determine the constants λ_j and μ_j such that

$$G_n \sim \frac{1}{\pi}\ln\left(n+\frac{3}{4}\right)+c_0+\frac{1}{\pi}\sum_{j=1}^{\infty}\frac{\lambda_j}{\left\{n^2+\frac{3}{2}n+\mu_j\right\}^{2j-1}}\,,\quad n\to\infty. \tag{1.13}$$

There are bounds of other types involved. For example, Alzer [4, Theorem 1] gave the upper and lower bounds for the Landau constants in terms of the digamma function $\psi=\Gamma'/\Gamma$, namely

$$c_0+\frac{1}{\pi}\psi(n+\alpha)<G_n\le c_0+\frac{1}{\pi}\psi(n+\beta),\ \ n\in\mathbb{N}_0,$$

with the best possible constants

$$\alpha=\frac{5}{4}\quad\text{and}\quad\beta=1.26621\,\ldots\,.$$

Based on the above result of Nemes [22], Chen [8] obtained the following full asymptotic expansion:

$$G_n \sim c_0+\frac{1}{\pi}\psi\left(n+\frac{5}{4}\right)+\sum_{k=1}^{\infty}\frac{q_k(h)}{(n+h)^k}\,,\quad n\to\infty \tag{1.14}$$

with the coefficients $q_k(h)$, $k\in\mathbb{N}$ given by

$$q_k(h)=(-1)^k\frac{B_k\left(\frac{5}{4}-h\right)}{k\pi}-g_k(h)\,,\quad k\in\mathbb{N}, \tag{1.15}$$

where $B_n(t)$ are the Bernoulli polynomials and $g_k(h)$ are given in (1.5).

Noting that [29, p. 804, (23.1.8)]

$$B_n(1-x)=(-1)^nB_n(x)\,,\ \ n\in\mathbb{N}_0\,,$$

we obtain

$$B_{2k-1}\left(\frac{1}{2}\right)=0\,,\qquad k\in\mathbb{N}.$$

We find from (1.15) that

$$q_{2k-1}\left(\frac{3}{4}\right)=\frac{-B_{2k-1}\left(\frac{1}{2}\right)}{(2k-1)\pi}-g_{2k-1}\left(\frac{3}{4}\right)=0,\ \ k\in\mathbb{N}, \tag{1.16}$$

$$q_{2k}\left(\frac{3}{4}\right)=\frac{B_{2k}\left(\frac{1}{2}\right)}{2k\pi}-g_{2k}\left(\frac{3}{4}\right)=0,\ \ k\in\mathbb{N}.$$

Noting that [1, p. 805, (23.1.21)]

$$B_n\left(\frac{1}{2}\right)=-(1-2^{-n})B_n\,,\ \ n\in\mathbb{N}_0\,,$$

the formula (1.16) can be written as

$$q_{2k}\left(\frac{3}{4}\right)=\frac{-(1-2^{1-2k})B_{2k}}{2k\pi}-g_{2k}\left(\frac{3}{4}\right),\ \ k\in\mathbb{N}. \tag{1.17}$$

The choice $h=\frac{3}{4}$ in (1.14) yields

$$G_n\sim\frac{1}{\pi}\psi\left(n+\frac{5}{4}\right)+c_0+\sum_{k=1}^{\infty}\frac{q_{2k}\left(\frac{3}{4}\right)}{\left(n+\frac{3}{4}\right)^{2k}}\ ,\ \ n\to\infty. \tag{1.18}$$

Using (1.17), we give the first few coefficients $q_{2k}\left(\frac{3}{4}\right)$ as follows:

$$\begin{gathered}q_2\left(\frac{3}{4}\right)=\frac{1}{64\pi},\qquad q_4\left(\frac{3}{4}\right)=-\frac{43}{8192\pi},\qquad q_6\left(\frac{3}{4}\right)=\frac{503}{131072\pi},\\ q_8\left(\frac{3}{4}\right)=-\frac{335891}{67108864\pi},\qquad q_{10}\left(\frac{3}{4}\right)=\frac{5575883}{536870912\pi},\\ q_{12}\left(\frac{3}{4}\right)=-\frac{2177397749}{68719476736\pi},\qquad q_{14}\left(\frac{3}{4}\right)=-\frac{147454921819}{1099511627776\pi}.\end{gathered} \tag{1.19}$$

Using the computer program MAPLE 11, we find, as $n\to\infty$,

$$\begin{aligned}G_n\sim{}&\frac{1}{\pi}\psi\left(n+\frac{5}{4}\right)+c_0\\&+\frac{1}{\pi}\left\{\left(\frac{\frac{1}{64}}{n^2+\frac{3}{2}n+\frac{115}{128}}+\frac{\frac{2175}{1048576}}{\left\{n^2+\frac{3}{2}n+\frac{14161}{11136}\right\}^3}\right.\right.\\&\left.\left.+\frac{\frac{1957143745}{498216206336}}{\left\{n^2+\frac{3}{2}n+\frac{23481771701867}{13076851646592}\right\}^5}\right)+\cdots\right\}.\end{aligned} \tag{1.20}$$

From a computational view point, the formula (1.20) is better than the formula (1.18). The second aim of the present paper is to determine the constants r_j and s_j such that

$$G_n\sim\frac{1}{\pi}\ln\left(n+\frac{5}{4}\right)+c_0+\frac{1}{\pi}\sum_{j=1}^{\infty}\frac{r_j}{\left(n^2+\frac{3}{2}n+s_j\right)^{2j-1}}\ ,\ \ n\to\infty. \tag{1.21}$$

We now define the sequence $(v_n)_{n\in\mathbb{N}}$ by

$$v_n = G_n - c_0 - \frac{1}{3\pi}\ln(n^3 + an^2 + bn + c). \tag{1.22}$$

We are interested in finding the values of the parameters $a, b,$ and c such that $(v_n)_{n\in\mathbb{N}}$ is the fastest sequence that converges to zero. This provides the best approximations of the form:

$$G_n \approx c_0 - \frac{1}{3\pi}\ln(n^3 + an^2 + bn + c)\ ,\ \ n \to \infty. \tag{1.23}$$

Our study is based on Lemma 1 below, which provides a method for measuring convergence speed. Based on the approximation formula (1.23), we establish a sharp double inequality. This is the last aim of the present paper.

The numerical values given in this paper have been calculated by using the computer program MAPLE 11.

2. LEMMAS

Let $\mathbb{R}$ be the set of all real numbers.

Lemma 1 ([30,31]). If the sequence $(\lambda_n)_{n\in\mathbb{N}}$ converges to zero and if there exists the following limit:

$$\lim_{n\to\infty} n^k(\lambda_n - \lambda_{n+1}) = l \in \mathbb{R},\ \ k > 1, \tag{2.1}$$

then

$$\lim_{n\to\infty} n^{k-1}\lambda_n = \frac{l}{k-1}.$$

Lemma 1 provides a method for measuring the speed of convergence.

Lemma 2 ([32]). The following continued fraction formula holds true:

$$\left[\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)}\right]^2 = \cfrac{4}{1+4n+\cfrac{1^2}{2+8n+\cfrac{3^2}{2+8n+\cfrac{5^2}{2+8n+\ddots}}}}. \tag{2.2}$$

In 1654, Lord William Brouncker found the remarkable continued fraction formula (2.2). Formula (2.2) was not published by Brouncker himself, but first appeared in [32]. Formula (2.2) follows, for example, from Entry 25 in Chapter 12 of Ramanujan's notebook [33], which gives a more general continued-fraction formula for quotients of gamma functions and has several proofs published by different authors. Very recently, Granath [18] derived the asymptotic expansions for the Landau constants and related inequalities by using Brouncker's continued fraction formula.

By (2.2), we have the following inequality [18, p. 742]:

$$\frac{4}{1+4n+\cfrac{1^2}{2+8n+\cfrac{3^2}{2+8n+\cfrac{5^2}{2+8n}}}} < \left[\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)}\right]^2$$
$$< \frac{4}{1+4n+\cfrac{1^2}{2+8n+\cfrac{3^2}{2+8n+\cfrac{5^2}{2+8n+\cfrac{7^2}{2+8n}}}}}, \quad n \in \mathbb{N},$$

that is

$$\frac{16(19+92n+96n^2+128n^3)}{105+704n+1920n^2+2048n^3+2048n^4} < \left[\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)}\right]^2$$
$$< \frac{4(789+2912n+6848n^2+4096n^3+4096n^4)}{945+6756n+18880n^2+32000n^3+20480n^4+16384n^5}, \qquad n \in \mathbb{N}. \tag{2.3}$$

Lemma 3. For x ≥ 1, the following double inequality holds:

$$\frac{3}{x}-\frac{3}{4x^2}+\frac{3}{32x^3}+\frac{3}{128x^4}-\frac{1011}{10240x^5}+\frac{867}{8192x^6}-\frac{63681}{917504x^7}$$
$$< \ln\frac{x^3+\frac{9}{4}x^2+\frac{119}{64}x+\frac{141}{256}}{(x-1)^3+\frac{9}{4}(x-1)^2+\frac{119}{64}(x-1)+\frac{141}{256}} \tag{2.4}$$
$$< \frac{3}{x}-\frac{3}{4x^2}+\frac{3}{32x^3}+\frac{3}{128x^4}-\frac{1011}{10240x^5}+\frac{867}{8192x^6}-\frac{63681}{917504x^7}$$
$$+\frac{17049}{524288x^8}.$$

Proof: The inequality (2.4) is obtained by considering the functions $f(x)$ and $g(x)$ defined, for $x \geq 1$, by

$$f(x) = \ln\frac{x^3+\frac{9}{4}x^2+\frac{119}{64}x+\frac{141}{256}}{(x-1)^3+\frac{9}{4}(x-1)^2+\frac{119}{64}(x-1)+\frac{141}{256}}$$
$$-\left(\frac{3}{x}-\frac{3}{4x^2}+\frac{3}{32x^3}+\frac{3}{128x^4}-\frac{1011}{10240x^5}+\frac{867}{8192x^6}-\frac{63681}{917504x^7}\right)$$

$$g(x) = \ln\frac{x^3+\frac{9}{4}x^2+\frac{119}{64}x+\frac{141}{256}}{(x-1)^3+\frac{9}{4}(x-1)^2+\frac{119}{64}(x-1)+\frac{141}{256}}$$
$$-\left(\frac{3}{x}-\frac{3}{4x^2}+\frac{3}{32x^3}+\frac{3}{128x^4}-\frac{1011}{10240x^5}+\frac{867}{8192x^6}-\frac{63681}{917504x^7}\right.$$
$$\left.+\frac{17049}{524288x^8}\right).$$

Differentiation yields

$$f'(x) = -\frac{3P_5(x-1)}{131072x^8(256x^3-192x^2+92x-15)(256x^3+576x^2+476x+141)}$$

and

$$g'(x) = \frac{3Q_5(x-1)}{131072x^9(256x^3-192x^2+92x-15)(256x^3+576x^2+476x+141)},$$

where

$$P_5(x) = 1712282791 + 7518123416x + 13267703152x^2 + 11572155392x^3 + 4810279936x^4 + 744882176x^5,$$

and

$$Q^5(x) = 609893303 + 2448840753x + 3817289816x^2 + 2744059536x^3 + 773132288x^4 + 31454208x^5.$$

Clearly, $f'(x) < 0$ and $g'(x) > 0$ hold for $x \geq 1$. Hence, the function $f(x)$ is strictly decreasing on $[1,\infty)$, $g(x)$ is strictly increasing on $[1,\infty)$, and we have

$$f(x) > \lim_{t\to\infty} f(t) = 0 \quad \text{and} \quad g(x) < \lim_{t\to\infty} g(t) = 0, \quad \text{for } x \geq 1.$$

The proof of Lemma 3 is complete.□

3. MAIN RESULTS

Theorem 1. As n → ∞, the Landau constants have the following asymptotic expansion:

$$\pi G_n - \pi c_0 - \ln\left(n+\frac{3}{4}\right) \sim \sum_{j=1}^{\infty} \frac{a_j}{\left\{\left(n+\frac{3}{4}\right)^2 + b_j\right\}^{2j-1}}, \tag{3.1}$$

where a_j and b_j are given by a pair of recurrence relations

$$a_j = -\pi g_{4j-2}\left(\frac{3}{4}\right) - \sum_{k=0}^{j-2} a_{k+1} b_{k+1}^{2j-2k-2} \binom{2j-2}{2j-2k-2}, \tag{3.2}$$

$$b_j = \frac{1}{(2j-1)a_j}\left\{\pi g_{4j}\left(\frac{3}{4}\right) - \sum_{k=0}^{j-2} a_{k+1} b_{k+1}^{2j-2k-1} \binom{2j-1}{2j-2k-1}\right\} \tag{3.3}$$

for $j \geq 2$, with $a_1 = \frac{11}{192}$ and $b_1 = \frac{1541}{7040}$. Here $g_{2k}\left(\frac{3}{4}\right)$ can be calculated using (1.5) or (1.7).

Proof: In view of (1.12), we can let

$$\pi G_n - \pi c_0 - \ln\left(n+\frac{3}{4}\right) \sim \sum_{j=1}^{\infty} \frac{a_j}{\left\{\left(n+\frac{3}{4}\right)^2 + b_j\right\}^{2j-1}}$$

as $n \to \infty$, where a_l and b_l are real numbers to be determined. It follows from (1.10) and (1.11) that

$$\pi G_n - \pi c_0 - \ln\left(n+\frac{3}{4}\right) \sim \sum_{j=1}^{\infty}\left(-\pi g_{2l}\left(\frac{3}{4}\right)\right)\left(n+\frac{3}{4}\right)^{-2l}, \tag{3.4}$$

where $g_{2l}\left(\frac{3}{4}\right)$ can be calculated using (1.5) or (1.7). Direct computation yields

$$\begin{aligned}
\sum_{j=1}^{\infty}\frac{a_j}{\left\{\left(n+\frac{3}{4}\right)^2+b_j\right\}^{2j-1}} &= \sum_{j=1}^{\infty}\frac{a_j}{\left(n+\frac{3}{4}\right)^{4j-2}}\left(1+\frac{b_j}{\left(n+\frac{3}{4}\right)^2}\right)^{-(2j-1)}\\
&= \sum_{j=1}^{\infty}\frac{a_j}{\left(n+\frac{3}{4}\right)^{4j-2}}\sum_{k=0}^{\infty}\binom{-(2j-1)}{k}\frac{b_j^k}{\left(n+\frac{3}{4}\right)^{2k}}\\
&= \sum_{j=1}^{\infty}\frac{a_j}{\left(n+\frac{3}{4}\right)^{4j-2}}\sum_{k=0}^{\infty}(-1)^k\binom{k+2j-2}{k}\frac{b_j^k}{\left(n+\frac{3}{4}\right)^{2k}}\\
&= \sum_{j=0}^{\infty}\frac{a_{j+1}}{\left(n+\frac{3}{4}\right)^{4j+2}}\sum_{k=0}^{\infty}(-1)^k\binom{k+2j}{k}\frac{b_{j+1}^k}{\left(n+\frac{3}{4}\right)^{2k}}\\
&= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}a_{k+1}(-1)^{j-k}\binom{j+k}{j-k}b_{k+1}^{j-k}\frac{1}{\left(n+\frac{3}{4}\right)^{2(j+k+1)}}\\
&= \sum_{l=1}^{\infty}\left\{\sum_{k=0}^{\left\lfloor\frac{l-1}{2}\right\rfloor}a_{k+1}(-1)^{l-1}b_{k+1}^{l-2k-1}\binom{l-1}{l-2k-1}\right\}\left(n+\frac{3}{4}\right)^{-2l}.
\end{aligned} \tag{3.5}$$

Equating coefficients of the term $\left(n+\frac{3}{4}\right)^{-2l}$, on the right sides of (3.4) and (3.5), we obtain

$$\sum_{k=0}^{\left\lfloor\frac{l-1}{2}\right\rfloor}a_{k+1}(-1)^{l-1}b_{k+1}^{l-2k-1}\binom{l-1}{l-2k-1} = -\pi g_{2l}\left(\frac{3}{4}\right),\ \ l \geq 1. \tag{3.6}$$

Setting $l = 2j+1$ and $l = 2j+2$ in (3.6), respectively, yields

$$\sum_{k=0}^{j}a_{k+1}b_{k+1}^{2j-2k}\binom{2j}{2j-2k} = -\pi g_{4j+2}\left(\frac{3}{4}\right),\ \ j \geq 0. \tag{3.7}$$

and

$$\sum_{k=0}^{j}a_{k+1}b_{k+1}^{2j-2k+1}\binom{2j+1}{2j-2k+1} = \pi g_{4j+4}\left(\frac{3}{4}\right),\ \ j \geq 0. \tag{3.8}$$

For $j = 0$, from (3.7) and (3.8) we obtain

$$a^1=-\pi g_2\left(\frac{3}{4}\right)=\beta_2=\frac{11}{192},\quad b_1=\frac{\pi g_4\left(\frac{3}{4}\right)}{a^1}=\frac{-\beta_4}{a^1}=\frac{\frac{1541}{122880}}{\frac{11}{192}}=\frac{1541}{7040};$$

and for $j\geq 1$ we have

$$\sum_{k=0}^{j-1}a_{k+1}b_{k+1}^{2j-2k}\binom{2j}{2j-2k}+a_{j+1}=-\pi g_{4j+2}\left(\frac{3}{4}\right),$$
$$\sum_{k=0}^{j-1}a_{k+1}b_{k+1}^{2j-2k+1}\binom{2j+1}{2j-2k+1}+a_{j+1}b_{j+1}(2j+1)=\pi g_{4j+4}\left(\frac{3}{4}\right).$$

We then obtain the recurrence relations (3.2) and (3.3).

The proof of Theorem 1 is complete. □

Here, we give explicit numerical values of some of the first terms of a_j and b_j by using (3.2) and (3.3). This shows how easily we can determine the constants a_j and b_j in (3.1).

$$a_1=\frac{11}{192},\quad b_1=\frac{1541}{7040}$$
$$a_2=-\pi g_6\left(\frac{3}{4}\right)-a_1b_1^2=\frac{89684299}{18166579200},\quad b_2=\frac{\pi g_8\left(\frac{3}{4}\right)-a_1b_1^3}{3a_2}=\frac{364000123403}{631377464960},$$
$$a_3=-\pi g_{10}\left(\frac{3}{4}\right)-a_1b_1^4-6a_2b_2^2=\frac{306469422974562781964487}{38451704669013855436800000},$$
$$b_3=\frac{\pi g_{12}\left(\frac{3}{4}\right)-a_1b_1^5-10a_2b_2^3}{5a_3}$$
$$=\frac{956149334232701916667332221848871663 9}{880415387512722114878118969110596160 0}.$$

Remark 1. The constants λ$_j$ and μ$_j$ in (1.13) are given by

$$\lambda_j=a_j\ ,\quad \mu_j=\frac{9}{16}+b_j\,.$$

The few constants λ$_j$ and μ$_j$ are

$$\lambda_1=\frac{11}{192},\quad \mu_1=\frac{5501}{7040},\qquad \lambda_2=\frac{89684299}{18166579200},\quad \mu_2=\frac{719149947443}{631377464960},$$
$$\lambda_3=\frac{306469422974562781964487}{38451704669013855436800000},$$
$$\mu_3=\frac{14513829897086081062862741419735820039}{8804153875127221148781189691105961600}.$$

We note that the values of λ_j and μ_j (for $j=1,2,3$) above are equal to the constants appearing in (1.12).

Following the same method as was used in the proof of Theorem 1, we can prove the following Theorem 2. We here omit the proof.

Theorem 2. As n → ∞, the Landau constants have the following asymptotic expansion:

$$\pi G_n - \pi c_0 - \ln\left(n+\frac{5}{4}\right) \sim \sum_{j=1}^{\infty} \frac{\alpha_j}{\left\{\left(n+\frac{3}{4}\right)^2 + \beta_j\right\}^{2j-1}}, \tag{3.9}$$

where α_j and β_j are given by a pair of recurrence relations

$$\alpha_j = \pi g_{4j-2}\left(\frac{3}{4}\right) - \sum_{k=0}^{j-2} \alpha_{k+1}\beta_{k+1}^{2j-2k-2}\binom{2j-2}{2j-2k-2}, \tag{3.10}$$

$$\beta_j = \frac{1}{(2j-1)\alpha_j}\left\{-\pi g_{4j}\left(\frac{3}{4}\right) - \sum_{k=0}^{j-2} \alpha_{k+1}\beta_{k+1}^{2j-2k-1}\binom{2j-1}{2j-2k-1}\right\} \tag{3.11}$$

for j ≥ 2, with $\alpha_1 = \frac{1}{64}$ and $\beta_1 = \frac{43}{128}$. Here $q_{2k}\left(\frac{3}{4}\right)$ can be calculated using (1.17).

Noting that (1.19), we here give explicit numerical values of some first terms of α_j and β_j by using (3.10) and (3.11).

$$\alpha_1 = \frac{1}{64}\ ,\quad \beta_1 = \frac{43}{128},$$

$$\alpha_2 = \pi q_6\left(\frac{3}{4}\right) - \alpha_1\beta_1^2 = \frac{2175}{1048576},\quad \beta_2 = -\frac{\pi q_8\left(\frac{3}{4}\right) + \alpha_1\beta_1^3}{3\alpha_2} = \frac{7897}{11136}\ ,$$

$$\alpha_3 = \pi q_{10}\left(\frac{3}{4}\right) - \alpha_1\beta_1^4 - 6\alpha_2\beta_2^2 = \frac{1957143745}{498216206336},$$

$$\beta_3 = -\frac{\pi q_{12}\left(\frac{3}{4}\right) + \alpha_1\beta_1^5 + 10\alpha_2\beta_2^3}{5\alpha_3} = \frac{16126042650659}{13076851646592}.$$

Remark 2. The constants r_j and s_j in (1.21) are given by

$$r_j = \alpha_j\ ,\quad s_j = \frac{9}{16} + \beta_j\ .$$

The first few constants r_j and s_j are

$$r_1 = \frac{1}{64},\quad s_1 = \frac{115}{128}, r_2 = \frac{2175}{1048576},\quad s_2 = \frac{14161}{11136}\ ,$$

$$r_3 = \frac{1957143745}{498216206336},\quad s_3 = \frac{23481771701867}{13076851646592}.$$

We note that the values of r_j and s_j (for $j = 1,2,3$) above are equal to the constants appearing in (1.20).

Theorem 3. Let the sequence $(v_n)_{n\in\mathbb{N}}$ be defined by 1.22). Suppose also that

$$a = \frac{9}{4},\qquad b = \frac{119}{64},\qquad c = \frac{141}{256}. \tag{3.12}$$

Then

$$\lim_{n\to\infty} n^5(v_n - v_{n+1}) = -\frac{39}{1280\pi} \quad \text{and} \quad \lim_{n\to\infty} n^4 v_n = -\frac{39}{5120\pi}. \tag{3.13}$$

The speed of convergence of the sequence $(v_n)_{n\in\mathbb{N}}$ is given by the order estimate $O(n^{-4})$ as $n \to \infty$.

Proof: Using the representation (see [18, p. 739], [2, p. 218])

$$G_n - G_{n-1} = \frac{1}{\pi}\left(\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)}\right)^2, \tag{3.14}$$

we obtain

$$v_n - v_{n+1} = -\frac{1}{\pi}\left(\frac{\left(n+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{(n+1)\Gamma(n+1)}\right)^2 + \frac{1}{3\pi}\ln\frac{(n+1)^3 + a(n+1)^2 + b(n+1) + c}{n^3 + an^2 + bn + c}.$$

We write the difference $v_n - v_{n+1}$ as the following power series in n^{-1}:

$$\begin{aligned} v_n - v_{n+1} &= \frac{9-4a}{12\pi n^2} + \frac{-115 + 32a - 64b + 32a^2}{96\pi n^3} \\ &+ \frac{609 - 128a + 384b - 384c - 192a^2 + 384ab - 128a^3}{384\pi n^4} \\ &+ \frac{1}{30720\pi n^5}\{-60021 + 10240a - 40960b + 20480a^2 + 61440c \\ &+ 20480a^3 - 61440ab + 40960ac + 20480b^2 + 10240a^4 \\ &- 40960a^2 b\} + O\left(\frac{1}{n^6}\right). \end{aligned} \tag{3.15}$$

According to Lemma 1, the three parameters a, b and c, which produce the fastest convergence of the sequence $(v_n)_{n\in\mathbb{N}}$ are given by (3.15)

$$\begin{cases} 9 - 4a = 0 \\ -115 + 32a - 64b + 32a^2 = 0 \\ 609 - 128a + 384b - 384c - 192a^2 + 384ab - 128a^3 = 0, \end{cases}$$

that is, by (3.12). We thus find that

$$v_n - v_{n+1} = -\frac{39}{1280\pi n^5} + O\left(\frac{1}{n^6}\right), \quad n \to \infty.$$

Finally, by using Lemma 1, we obtain the desired assertion (3.13).
The proof of Theorem 3 is complete. □

We then find that

$$G_n \approx c_0 + \frac{1}{3\pi}\ln\left(n^3 + \frac{9}{4}n^2 + \frac{119}{64}n + \frac{141}{256}\right), \quad n \to \infty \tag{3.16}$$

is the best approximation among all approximations given by (1.23).

The formula (3.16) has motivated us to propose the following question: What is the largest number α, and what is the smallest number β such that the following inequality:

$$c_0+\frac{1}{3\pi}\ln\left(n^3+\frac{9}{4}n^2+\frac{119}{64}n+\alpha\right)<G_n<c_0+\frac{1}{3\pi}\ln\left(n^3+\frac{9}{4}n^2+\frac{119}{64}n+\beta\right)$$

holds true for all n ∈ ℕ ? Theorem 4 answers this question.

Theorem 4. For all n ∈ $\mathbb{N}_0$, the following sharp bounds hold true:

$$\begin{aligned}c_0+\frac{1}{3\pi}\ln\left(n^3+\frac{9}{4}n^2+\frac{119}{64}n+e^{3\pi(1-c_0)}\right)&\le G_n\\ <c_0+\frac{1}{3\pi}\ln\left(n^3+\frac{9}{4}n^2+\frac{119}{64}n+\frac{141}{256}\right),&\end{aligned}\tag{3.17}$$

where the constants

$$e^{3\pi(1-c_0)}=0.53545672\ldots\ \ and\ \ \frac{141}{256}=0.55078125\ldots$$

are the best possible, in the sense that the first (the second) constant cannot be replaced by a larger (smaller) value, respectively.

Proof: The upper bound in (3.17) is obtained by considering the sequence $(x_n)_{n\in\mathbb{N}}$ defined by

$$x_n=G_n-c_0-\frac{1}{3\pi}\ln\left(n^3+\frac{9}{4}n^2+\frac{119}{64}n+\frac{141}{256}\right),\ \ n\in\mathbb{N}_0.$$

Using (3.14), (2.3) and (2.4), we obtain that for $n\in\mathbb{N}$,

$$\begin{aligned}x_n-x_{n-1}&=\frac{1}{\pi}\left(\frac{\Gamma\left(n+\left(\frac{1}{2}\right)\right)}{\Gamma(n+1)}\right)^2-\frac{1}{3\pi}\ln\left(\frac{n^3+\frac{9}{4}n^2+\frac{119}{64}n+\frac{141}{256}}{(n-1)^3+\frac{9}{4}(n-1)^2+\frac{119}{64}(n-1)+\frac{141}{256}}\right)\\ &>\frac{1}{\pi}\frac{16(19+92n+96n^2+128n^3)}{105+704n+1920n^2+2048n^3+2048n^4}\\ &\quad-\frac{1}{3\pi}\left(\frac{3}{n}-\frac{3}{4n^2}+\frac{3}{32n^3}+\frac{3}{128n^4}-\frac{1011}{10240n^5}+\frac{867}{8192n^6}-\frac{63681}{917504n^7}\right.\\ &\quad\left.+\frac{17049}{524288n^8}\right)\\ &=\frac{P_7(n-1)}{18350080\pi n^8(105+704n+1920n^2+2048n^3+2048n^4)},\end{aligned}$$

where

$$\begin{aligned}P_7(n)=778168475&+7173768060n+22843141568n^2+38304073984n^3\\ &+38122039296n^4+22870401024n^5+7729053696n^6\\ &+1145044992n^7.\end{aligned}$$

Clearly, $x_n>x_{n-1}$ holds for $n\in\mathbb{N}$. Therefore, the sequence $(x_n)_{n\in\mathbb{N}}$ is strictly increasing, and we have

$$x_n<\lim_{m\to\infty}x_m=0\,,\ \ n\in\mathbb{N}.$$

This means that the right-hand side of (3.17) holds for $n \in \mathbb{N}_0$.

The right-hand side of (3.17) can be written as

$$\frac{1}{\frac{192\pi}{11}\left[G_n - \frac{1}{\pi}\ln\left(n+\frac{3}{4}\right) - c_0\right]} - n^2 - \frac{3}{2}n < \frac{5501}{7040}.$$

Using (1.10), we find

$$\lim_{n\to\infty}\left(\frac{1}{\frac{192\pi}{11}\left[G_n - \frac{1}{\pi}\ln\left(n+\frac{3}{4}\right) - c_0\right]} - n^2 - \frac{3}{2}n\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{\frac{192\pi}{11}\frac{1}{\pi}\left[\frac{11}{192\left(n+\frac{3}{4}\right)^2} - \frac{1541}{122880\left(n+\frac{3}{4}\right)^4} + O\left(\frac{1}{\left(n+\frac{3}{4}\right)^6}\right)\right]} - n^2 - \frac{3}{2}n\right)$$

$$= \lim_{n\to\infty}\left(\frac{5501}{7040} + O\left(\frac{1}{n^2}\right)\right) = \frac{5501}{7040}.$$

Hence, the right-hand side of (3.17) holds for $n \in \mathbb{N}_0$, the constant $\frac{5501}{7040}$ is the best possible.

The lower bound in (3.17) is obtained by considering the sequence $(y_n)_{n\in\mathbb{N}_0}$ defined by

$$y_n = G_n - c_0 - \frac{1}{3\pi}\ln\left(n^3 + \frac{9}{4}n^2 + \frac{119}{64}n + a\right),$$

where

$$a = e^{3\pi(1-c_0)} = 0.53545672\ldots.$$

Direct computation would yield

$$\left[c_0 + \frac{1}{3\pi}\ln\left(n^3 + \frac{9}{4}n^2 + \frac{119}{64}n + e^{3\pi(1-c_0)}\right)\right]_{n=0} = 1,\ \ G_0 = 1.$$

Hence, for $n = 0$, the equal sign on the left-hand side of Theorem 4 holds.

Using (3.14), (2.3) and (2.4), and noting that $a < 0.54$, we obtain that for $n \geq 3$,

$$
\begin{aligned}
y_n - y_{n-1} &= \frac{1}{\pi}\left(\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)}\right)^2 - \frac{1}{3\pi}\ln\left(\frac{n^3+\frac{9}{4}n^2+\frac{119}{64}n+a}{(n-1)^3+\frac{9}{4}(n-1)^2+\frac{119}{64}(n-1)+a}\right) \\
&= \frac{1}{\pi}\left(\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)}\right)^2 - \frac{1}{3\pi}\ln\left(1+\frac{3(64n^2+32n+13)}{64n^3-48n^2+23n-39+64a}\right) \qquad (3.18) \\
&< \frac{1}{\pi}\frac{4(789+2912n+6848n^2+4096n^3+4096n^4)}{945+6756n+18880n^2+32000n^3+20480n^4+16384n^5} \\
&\quad - \frac{1}{3\pi}\ln\left(1+\frac{3(64n^2+32n+13)}{64n^3-48n^2+23n-39+64\cdot 0.54}\right) =: \frac{1}{\pi}h(n),
\end{aligned}
$$

where

$$
\begin{aligned}
h(x) = {}& \frac{4(789+2912x+6848x^2+4096x^3+4096x^4)}{945+6756x+18880x^2+32000x^3+20480x^4+16384x^5} \\
& - \frac{1}{3}\ln\left(1+\frac{3(64x^2+32x+13)}{64x^3-48x^2+23x-39+64\cdot 0.54}\right).
\end{aligned}
$$

Differentiation yields

$$
h'(x) = \frac{3P_{11}(x-3)}{P_{16}(x)},
$$

where

$$
\begin{aligned}
P_{11}(x) = {}& 70417219569001425 + 1598504448371650200x \\
& + 4466664707687718048x^2 + 5913280509562446848x^3 \\
& + 4712671890810261504x^4 + 2485500067679272960x^5 \\
& + 904310091021287424x^6 + 229655127584145408x^7 \\
& + 40182004850884608x^8 + 4634937445580800x^9 \\
& + 318257076633600x^{10} + 9878424780800x^{11}, \\
P_{16}(x) = {}& (1600x^3+3600x^2+2975x+864)(1600x^3-1200x^2+575x-111) \\
& \times (945+6756x+18880x^2+32000x^3+20480x^4+16384x^5)^2.
\end{aligned}
$$

Clearly, $h'(x) > 0$ holds for $x \geq 3$. Hence, the function $h(x)$ is strictly increasing for $x \geq 3$, and we have

$$
h(x) < \lim_{t\to\infty} h(t) = 0 \quad \text{for} \quad x \geq 3.
$$

We see from (3.18) that, $y_n < y_{n-1}$ holds for all $n \geq 3$. Direct computation would yield

$$
y_1 = 0.00008688\ldots, \quad y_2 = 0.00003860\ldots.
$$

Hence, the sequence $(y_n)_{n\in\mathbb{N}}$ is strictly decreasing for $n \in \mathbb{N}$, and we have

$$
y_n > \lim_{m\to\infty} y_m = 0 \quad \text{for} \quad n \in \mathbb{N}.
$$

Therefore, the left-hand side of (3.17) holds for all $n \in \mathbb{N}_0$. Noting that the equal sign on the left-hand side of (3.17) holds for $n = 0$, we see that the constant $e^{3\pi(1-c_0)}$ is the best possible. The proof of Theorem 4 is complete. □

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