ORIGINAL PAPER

GENERALIZATION OF FIBONACCI POLYNOMIALS

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Abstract. *Fibonacci polynomials are special cases of Chebyshev polynomials and have been studied on a more advanced level by many mathematicians. Fibonacci polynomials are defined by* $f_{n+1}(x) = xf_n(x) + f_{n-1}(x), n \geq 1$ with $f_0(x) = 0,\ f_1(x) = 1$. *The Fibonacci polynomials are of great importance in the study of many subjects, such as algebra, geometry, and number theory itself. Obviously, they have a deep relationship with the famous Fibonacci sequence. That is,* $f_n(1) = F_n$, *where* F_n *is a Fibonacci number. In this paper, we introduce the generalization of Fibonacci polynomials by changing the initial terms with preservation of recurrence relation. Furthermore, some properties and identities of generalized Fibonacci polynomials are established and derived by standard methods.*

Keywords: *Fibonacci polynomials; Generalized Fibonacci polynomials; Binet's Formula; Fibonacci matrix.*

Mathematics Subject Classification: *11B37; 11B39; 11B83.*

1. INTRODUCTION

Fibonacci polynomials are special cases of Chebyshev polynomials and have been studied on a more advanced level by many mathematicians. The Fibonacci polynomials appear as the elements of the Q matrix. Some identities can be derived for Fibonacci polynomials using matrix theory, as done by Hayes, Richard A.

In 1881, Catlan [1] considered a related set of polynomials that satisfy the recurrence relation

$$y_{k+2}(x) = xy_{k+1}(x) + y_k(x). \tag{1.1}$$

In 1919-20, Jacosthal [2] investigated the solution of the relation:

$$z_{k+2}(x) = z_{k+1}(x) + xz_k(x) \text{ and } z_0(x) = 0,\ z_1(x) = 1, \tag{1.2}$$

and he first gave the name "Fibonacci polynomials".

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Byrd, P. F. [3] defined Fibonacci polynomials $\{\phi_k(x)\}$ by recurrence relation:

$$\phi_{k+2}(x) = 2x\phi_{k+1}(x) + \phi_k(x) \text{ and } \phi_0(x) = 0,\ \phi_1(x) = 1. \tag{1.3}$$

Basin, S. L. [4] shows that the Q matrix generates a set of Fibonacci Polynomials satisfying the recurrence relation:

$$f_n(x) = xf_{n-1}(x) + f_{n-2}(x) \text{ and } f_1(x) = 1,\ f_2(x) = x. \tag{1.4}$$

He derives the explicit forms and generating function by the matrix method. Also, Hayes [5] derives many identities by the matrix method.

Swami, M. N. S. [6] and Hoggatt, V. E., Jr. [7] almost simultaneously defined the Fibonacci polynomials $f_n(x)$ by

$$f_{n+1}(x) = xf_n(x) + f_{n-1}(x) \text{ and } f_0(x) = 0,\ f_1(x) = 1. \tag{1.5}$$

Equation (1.4) is now the accepted form of Fibonacci polynomials. The first few polynomials of (1.4) are

$$f_1(x) = 1, \quad f_2(x) = x, \quad f_3(x) = x^2 + 1, \quad f_4(x) = x^3 + 2x,$$

$$f_5(x) = x^4 + 3x^2 + 1, \quad f_6(x) = x^5 + 4x^3 + 3x.$$

The Fibonacci polynomials have a deep relationship with the famous Fibonacci numbers F_n. That is,

$$f_n(1) = F_n, \tag{1.6}$$

where F_n is Fibonacci numbers

The Fibonacci polynomials possess many fascinating properties, which have been studied in [1,2,4-17],

The convolved Fibonacci polynomials (CFPs) are considered generalizations of the standard Fibonacci polynomials [18]. New families of Generalized Fibonacci polynomials and Generalized Lucas polynomials are defined, and some elegant properties of these families. Also, the relationships between the family of the generalized k-Fibonacci polynomials and the known generalized Fibonacci polynomials are described [19].

The relationship between two different generalized Fibonacci polynomials is established, and generalized classical results are derived using matrices [20]. The accumulation points of the zero set of the polynomial family generated from complex polynomial seeds were identified [21]. Few scholars generalized Fibonacci polynomials by changing both the recurrence relation and initial conditions and derived some fundamental properties of generalized polynomials, such as explicit sum formula, sum of first n terms, sum of first n terms with (odd or even) indices, and generalized identity by Binet's formula and generating function only [22].

Many scholars have generalized Fibonacci polynomials in a number of ways. Mainly, two certain recursive schemes are observed, which are associated with the generalization of Fibonacci polynomials: (1) By changing the recurrence relation while preserving the initial terms, (2) By changing the initial terms, but the recurrence relation is preserved.

In this paper, we introduce a generalization of Fibonacci polynomials by changing the initial terms, but the recurrence relation is preserved. Also state and prove some properties of such generalized Fibonacci polynomials by generating function, Binet's form, and induction methods.

2. GENERALIZATION OF FIBONACCI POLYNOMIALS

Generalization of Fibonacci polynomials can be proposed by following two schemes [23]:

First Scheme

$$g_{n+2}(x) = a\,x\,g_{n+1}(x) + b\,g_n(x),\ n \geq 0 \text{ and } g_0(x) = 0,\ g_1(x) = 1, \tag{2.1}$$

where a and b are arbitrary integers.

Second Scheme

$$h_{n+2}(x) = x\,h_{n+1}(x) + h_n(x),\ n \geq 0 \text{ and } h_0(x) = a,\ h_1(x) = bx, \tag{2.2}$$

where a and b are arbitrary integers.

This can be written as:

$$h_{n+2}(x) = b\,x\,f_{n+2}(x) + b\,f_{n+1}(x),\ n \geq 0 \text{ and } f_0(x) = 0,\ f_1(x) = 1. \tag{2.3}$$

Now introduce some preliminaries for generalized Fibonacci polynomials (2.2).

3. PRELIMINARIES OF GENERALIZED FIBONACCI POLYNOMIALS

A few polynomials of the Second scheme (2.2) [23] are as under

Table 1. Polynomials of the first scheme.

n	$h_n(x)$
0	a
1	bx
2	$bx^2 + a$
3	$bx^3 + (a+b)x$
4	$bx^4 + (a+2b)x^2 + a$
5	$bx^5 + (a+3b)x^3 + (2a+b)x$
6	$bx^6 + (a+4b)x^4 + (3a+3b)x^2 + a$

If a=1 and b=1, then conventional Fibonacci polynomials are obtained, i.e.,

$$h_n(x) = f_{n+1}(x). \tag{3.1}$$

If a=2 and b=1, then conventional Lucas polynomials are obtained, i.e.,

$$h_n(x) = l_n(x). \tag{3.2}$$

Generating Function: The generating function is the power series

$$H(t) = \sum_{n=0}^{\infty} h_n(x) t^n. \tag{3.3}$$

To find a closed form for H(t), observe that

$$t\,H(t) = \sum_{n=1}^{\infty} h_{n-1}(x) t^n \quad \text{and} \quad t^2\,h(t) = \sum_{n=2}^{\infty} h_{n-2}(x) t^n.$$

Thus,

$$\begin{aligned} &H(t) - xt\,H(t) - t^2\,H(t) \\ &= (h_0(x) + t\,h_1(x) + t^2\,h_2(x)) - x(t\,h_0(x) + t^2\,h_1(x)) - t^2\,h_0(x) \\ &+ \sum_{n=3}^{\infty} \left\{ h_n(x) - xh_{n-1}(x) - h_{n-2}(x) \right\} t^n = a - axt + btx. \end{aligned}$$

The closed form of the generating function for generalized Fibonacci polynomials is

$$H(t) = \frac{a - axt + btx}{1 - xt - t^2}. \tag{3.4}$$

Explicit Sum formula: $h_n(x)$ is a polynomial of degree n-1. Its explicit sum formula is given by

$$h_n(x) = a \sum_{k=0}^{[\frac{n}{2}]} \binom{n-k}{k} x^{n-2k} + (b-a) \sum_{k=0}^{[\frac{n-1}{2}]} \binom{n-k-1}{k} x^{n-2k}, \tag{3.5}$$

where $\binom{n}{m}$ is a binomial coefficient and [p] is defined as the greatest integer less than or equal to p.

Binet’s formula: Binet’s formula is given by

$$h_n(x) = \frac{1}{\alpha - \beta} \left\{ (bx - a\beta)\,\alpha^n + (a\alpha - bx)\,\beta^n \right\}, \tag{3.6}$$

where α and β are the roots of the characteristic equation $t^2 - x\,t - 1 = 0$ and given by

$$\alpha = \frac{x+\sqrt{x^2+4}}{2},\ \beta = \frac{x-\sqrt{x^2+4}}{2}. \tag{3.7}$$

Here

$$\alpha+\beta = x,\ \ \alpha-\beta = \sqrt{x^2+4} \text{ and } \alpha.\beta = -1. \tag{3.8}$$

Fibonacci matrix: Define the matrix

$$A = A(x) = \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.9}$$

By induction,

$$A^n(x) = \begin{pmatrix} f_{n+1}(x) & f_n(x) \\ f_n(x) & f_{n-1}(x) \end{pmatrix}, \tag{3.10}$$

where $f_n(x)$ is the nth Fibonacci polynomial.
Now define the matrix

$$C = C(x) = \begin{pmatrix} bx^2+a & bx \\ bx & a \end{pmatrix} = \begin{pmatrix} h_2(x) & h_1(x) \\ h_1(x) & h_0(x) \end{pmatrix}. \tag{3.11}$$

Again, by induction, we get

$$CA^n = \begin{pmatrix} h_{n+2}(x) & h_{n+1}(x) \\ h_{n+1}(x) & h_n(x) \end{pmatrix}. \tag{3.12}$$

4. SOME PROPERTIES OF GENERALIZED FIBONACCI POLYNOMIALS

Now state and prove some properties [23].

Sum formulae: By rearranging the terms, multiplying by a suitable multiplier, and adding to get sum identities:

$$\begin{aligned} & x^n h_1(x) + x^{n-1}h_2(x) + x^{n-2}h_3(x) + \cdots + xh_n(x) \\ & = x\,h_{n+2}(x) - x^{n+1}h_2(x) = x\,h_{n+2}(x) - x^{n+1}(bx^2+a). \end{aligned} \tag{4.1}$$

i.e.,

$$\sum_{k=1}^{n} x^{n-k+1} h_k(x) = x\, h_{n+2}(x) - x^{n+1}(bx^2 + a).$$

$$\mathrm{h}_1(x) + h_3(x) + h_5(x) + \cdots + h_{2n-1}(x) = \frac{h_{2n}(x) - a}{x}. \tag{4.2}$$

$$\mathrm{h}_2(x) + h_4(x) + h_6(x) + \cdots + h_{2n}(x) = \frac{h_{2n+1}(x) - bx}{x}. \tag{4.3}$$

$$\begin{aligned} & x\left\{h_1(x) + h_2(x) + h_3(x) + \cdots + h_n(x)\right\} \\ & = (ax - bx - a) + h_{n+1}(x) + h_n(x) \end{aligned} \tag{4.4}$$

i.e.,

$$x\sum_{k=1}^{n} h_k(x) = (ax - bx - a) + h_{n+1}(x) + h_n(x).$$

Proof: Since

$$\sum_{k=1}^{n} y^k = \frac{1 - y^{n+1}}{1 - y} \quad \text{and} \quad (1-\alpha)(1-\beta) = -x, \tag{4.5}$$

Now

$$\begin{aligned} \sum_{k=1}^{n} h_k(x) &= \frac{1}{\alpha - \beta} h_n(x) \\ &= \frac{1}{\alpha - \beta}\left\{(bx - a\beta)\sum_{k=1}^{n}\alpha^k + (a\alpha - bx)\sum_{k=1}^{n}\beta^k\right\} \\ &= \frac{1}{\alpha - \beta}\left\{(bx - a\beta)\frac{(1-\alpha^{n+1})}{1-\alpha} + (a\alpha - bx)\frac{(1-\beta^{n+1})}{1-\beta}\right\} \qquad \text{(by 4.5)} \\ &= \frac{1}{(\alpha-\beta)(1-\alpha)(1-\beta)}\left\{(bx - a\beta)(1-\beta)(1-\alpha^{n+1}) + (a\alpha - bx)(1-\alpha)(1-\beta^{n+1})\right\} \\ &= \frac{-1}{x(\alpha - \beta)}\left\{\sqrt{x^2+4}(a - ax + bx) + (bx - a\beta)(1-\beta)\alpha^{n+1}) + (a\alpha - bx)(1-\alpha)\,\beta^{n+1})\right\} \text{ (by 4.5)} \\ &= \frac{1}{x}\left\{(-a + ax - bx) + h_{n+1}(x) + h_n(x)\right\}. \end{aligned}$$

Thus,

$$x\sum_{k=1}^{n} h_k(x) = (ax - bx - a) + h_{n+1}(x) + h_n(x).$$

Sums involving binomial coefficients: State sums involving the binomial coefficients. The proofs can be given from binomial expansion formulas and Binet's formula.

$$\mathrm{h}_{2n}(x)=\sum_{k=0}^{n}\binom{n}{k}x^k h_k(x). \tag{4.6}$$

$$\mathrm{h}_{4n+2}(x)=x^{2n+1}\binom{2n+1}{0}h_{2n+1}(x)+x^{2n}\binom{2n+1}{1}h_{2n}(x)+\cdots+\binom{2n+1}{2n+1}h_0(x). \tag{4.7}$$

$$\mathrm{h}_{4n}(x)=x^{2n}\binom{2n}{0}h_{2n}(x)+x^{2n-1}\binom{2n}{1}h_{2n-1}(x)+\cdots+\binom{2n}{2n}h_0(x). \tag{4.8}$$

$$\mathrm{h}_{p+n}(x)=\binom{n}{0}h_{p-n}(x)+x\binom{n}{1}h_{p-n+1}(x)+\cdots+x^n\binom{n}{n}h_p(x). \tag{4.9}$$

i.e.,

$$h_{p+n}(x)=\sum_{k=0}^{n}\binom{n}{k}x^k h_{p-n+k}(x).$$

$$\mathrm{h}_{p+2n}(x)=x^n\binom{n}{0}h_{p+n}(x)+x^{n-1}\binom{n}{1}h_{p+n-1}(x)+\cdots+\binom{n}{n}h_p(x). \tag{4.10}$$

i.e.,

$$h_{p+2n}(x)=\sum_{k=0}^{n}\binom{n}{k}x^{n-k}h_{p+n-k}(x).$$

Some standard identities: State and prove some identities like Catalan's identity, d'Occagne's identity, and Cassini's identity [23].

The proofs can be given from Binet's formula or the Fibonacci matrix.

$$\mathrm{h}_n^2(x)-h_{n+r}(x).h_{n-r}(x)=(-1)^{n-r+1}.f_r^{\,2}(x). \tag{4.11}$$

Proof: Using Binet's formula (3.6) in LHS, we have

$$\begin{aligned}&\mathrm{h}_n^2(x)-h_{n+r}(x).h_{n-r}(x)\\&=\frac{1}{(\alpha-\beta)^2}\left\{(bx-a\beta)\alpha^n+(a\alpha-bx)\beta^n\right\}^2\\&-\frac{1}{(\alpha-\beta)^2}\left\{(bx-a\beta)\alpha^{n-r}+(a\alpha-bx)\beta^{n-r}\right\}\left\{(bx-a\beta)\alpha^{n+r}+(a\alpha-bx)\beta^{n+r}\right\},\end{aligned}$$

$$h_n^2(x) - h_{n+r}(x).h_{n-r}(x) = \frac{(bx - a\beta)(a\alpha - bx)(-1)^{n-r+1}}{(\alpha-\beta)^2}(\alpha^r - \beta^r)^2$$
$$= (-1)^{n-r+1}(abx^2 + a^2 - b^2x^2)\, f_r^2(\text{x}). \quad \text{(by binet's formula)}$$

$$h_{m+1}(x)\, h_n(x) - h_m(x)\, h_{n+1}(x) = (-1)^{m+1}(abx^2 + a^2 - b^2x^2)h_{n-m}(x). \tag{4.12}$$
$$h_{n+1}(x).h_{n-1}(x) - h_n^2(x) = (-1)^{n-1}(abx^2 + a^2 - b^2x^2). \tag{4.13}$$

$$h_{n+2}^2(x) - h_n^2(x) = x.\left\{b^2x^2 f_{2n+2}(x) + 2abxf_{2n+1}(x) + a^2 f_{2n}(x)\right\}.$$
or (4.14)
$$h_{n+2}^2(x) - h_n^2(x) = ah_{n+1}(x)\left\{(a+bx^2)\, l_n(x) + bx\, l_{n-1}(x)\right\}.$$

$$h_n^2(x) + h_{n+1}^2(x) = b^2x^2 f_{2n+1}(x) + 2abxf_{2n}(x) + a^2 f_{2n-1}(x). \tag{4.15}$$

$$bx\, h_{m+n}(x) + a\, h_{m+n-1}(x) = h_{m+n}(x). \tag{4.16}$$

Proof: Define the matrix $A = A(x) = \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix}$.

By induction, we get $A^n(x) = \begin{pmatrix} f_{n+1}(x) & f_n(x) \\ f_n(x) & f_{n-1}(x) \end{pmatrix}$, where $f_n(x)$ nth Fibonacci polynomial is. Now by (3.11)

$$C = C(x) = \begin{pmatrix} bx^2 + a & bx \\ bx & a \end{pmatrix}$$

Since $A^{m+n} = A^m A^n$, we have

$$\begin{pmatrix} f_{m+n+1}(x) & f_{m+n}(x) \\ f_{m+n}(x) & f_{m+n-1}(x) \end{pmatrix} = \begin{pmatrix} f_{m+1}(x) & f_m(x) \\ f_m(x) & f_{m-1}(x) \end{pmatrix} . \begin{pmatrix} f_{n+1}(x) & f_n(x) \\ f_n(x) & f_{n-1}(x) \end{pmatrix}$$

$$= \begin{pmatrix} f_{m+1}(x) f_{n+1}(x) + f_m(x) f_n(x) & f_{m+1}(x) f_n(x) + f_m(x) f_{n-1}(x) \\ f_m(x) f_{n+1}(x) + f_{m-1}(x) f_n(x) & f_m(x) f_n(x) + f_{m-1}(x) f_{n-1}(x) \end{pmatrix}.$$

Now,

$$CA^{m+n} = \begin{pmatrix} bx^2 + a & bx \\ bx & a \end{pmatrix} \begin{pmatrix} f_{m+1}(x) f_{n+1}(x) + f_m(x) f_n(x) & f_{m+1}(x) f_n(x) + f_m(x) f_{n-1}(x) \\ f_m(x) f_{n+1}(x) + f_{m-1}(x) f_n(x) & f_m(x) f_n(x) + f_{m-1}(x) f_{n-1}(x) \end{pmatrix}.$$

Using matrix product, the Fibonacci polynomial identity, and then equating elements of both matrices, to get the required identity.

If a=2, b=1 in the second scheme (2.2), then Lucas polynomials are obtained. It is the most suitable example for the generalization of Fibonacci polynomials.

5. CONCLUSIONS

In this paper, a class of generalized Fibonacci polynomials is developed by modifying the initial conditions while preserving the classical second-order linear recurrence relation. Unlike many earlier generalizations that alter the recurrence coefficients, the present approach retains the original algebraic structure and introduces generalization only through parametric initial values.

The recurrence relation remains unchanged; the fundamental framework of Fibonacci polynomials is preserved. As a result, the generating function, explicit summation formula, Binet-type closed form, and matrix representation are obtained naturally and consistently within the same structural setting. Furthermore, classical identities such as Cassini's identity, Catalan's identity, and d'Ocagne's identity extended directly to this generalized family without modifying their essential form. Thus, compared with previous approaches, it provides a unified and structurally stable parametric generalization that preserves classical properties while broadening the theoretical scope of Fibonacci polynomials.

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