ORIGINAL PAPER

FRENET-SERRET FORMULAS AND FRENET ELEMENTS OF THE n-DIMENSIONAL QUATERNIONIC AND SPATIAL QUATERNIONIC SPACES

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Abstract. *Four-dimensional number systems are of great importance in physics, robotics, and computer-aided engineering. The quaternions give a chance to make examinations on 4-dimensional structures and also 3-dimensional calculations as spatial quaternions. The major goal of this research is to extend the Frenet elements for the Frenet-Serret n-vectors for spatial quaternionic and quaternionic curves constructed in $\mathbb{H}^n$ using real variable quaternion-valued vector functions. For this, the mathematical formulations of the Frenet elements in $\mathbb{H}_P^n$ are evaluated, and the results in $\mathbb{H}^n$ are presented by using them. Finally, exemplifications of spatial quaternion and quaternion curves with n-vectors are provided to support the conclusions.*

Keywords: *Frenet-Serret formulas; n-dimensional quaternionic space; quaternion-valued functions.*

Mathematics Subject Classification: *53A04; 53A25; 11R52; 53A40.*

1. INTRODUCTION

Quaternions have a wide range of applications across several fields of research [1–4]. Resources can be utilised for the basics of quaternion calculus. Also, [5–8] resources should be examined for more information in the fields of computer graphics and physics.

While traditional quaternionic Frenet-Serret frames are highly effective for modelling the kinematics and spatial rotations of a single rigid body in 3 or 4-dimensional spaces, extending these concepts to spaces where $n > 4$ presents a unique geometric advantage. In physical and geometric contexts, the n-dimensional quaternionic space $\mathbb{H}^n$ provides a unified mathematical framework for modeling complex, multi-agent systems, such as swarm robotics, cooperative unmanned aerial vehicle trajectories, or robotic manipulators, [9]. In these scenarios, the simultaneous orientation and position tracking of n distinct agents can be elegantly represented and analyzed as a single curve in $\mathbb{H}^n$. Furthermore, existing generalizations of Frenet-Serret formulas in higher-dimensional spaces typically rely on standard Euclidean or semi-Euclidean metrics (e.g., Aléssio, 2012 [10]). While such global metrics are mathematically profound, they collapse all dimensions, reducing the geometric properties of the entire space to a single scalar value. In contrast, the methodology proposed in this paper utilizes a componentwise metric. This choice decouples the inner-product operations across the n-subspaces, allowing for the independent yet simultaneous evaluation of spatial transformations. Moreover, computing higher-order curvatures in $\mathbb{R}^n$ using standard

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metrics leads to significant computational complexity, often requiring large-scale matrix operations as the dimension increases. Our componentwise quaternionic approach mitigates this computational burden while preserving the distinct geometric characteristics of each constituent curve.

The previously published paper [11] discusses the infrastructure of the operations to be employed in this study. For this reason, the essential concepts in [11] and [12] for the definitions and theorems that will be used while examining the curves on the n-dimensional quaternionic space will be addressed, in addition to the preliminaries.

Remark 1. In this respect, basic definitions, the componentwise multiplication on $\mathbb{R}^n$, and the componentwise metric that are defined are significant.

Component to component multiplication $\boxtimes_{\mathbb{R}^n}: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ for two quaternions n-vectors is defined as,

$$\vec{A} \boxtimes_{\mathbb{R}^n} \vec{B} = (a_1, a_2, \dots, a_n) \boxtimes_{\mathbb{R}^n} (b_1, b_2, \dots, b_n) = (a_1 b_1, a_2 b_2, \dots, a_n b_n) \tag{1}$$

for all $\vec{A} = (a_i), \vec{B} = (b_i) \in \mathbb{R}^n$.

On $\mathbb{R}^n$ ($\mathbb{R}^n$-module), the function $\|.\|: \mathbb{R}^n \to \mathbb{R}^n$,

$$\|\vec{A}\| = \sqrt{\vec{A} \boxtimes_{\mathbb{R}^n} \vec{A}} = \left(\sqrt{a_1^2}, \sqrt{a_2^2}, \dots, \sqrt{a_n^2}\right) = (|a_1|, |a_2|, \dots, |a_n|) \tag{2}$$

for all $\vec{A} = (a_i) \in \mathbb{R}^n$ defines a norm. With this norm function,

$$d(\vec{A}, \vec{B}) = \|\vec{A} - \vec{B}\| \tag{3}$$

defines a metric, called the componentwise metric for $\vec{A}, \vec{B} \in \mathbb{R}^n$. The proof of this metric can be reviewed from Theorem 3.2 of [11].

2. PRELIMINARIES

In this section, the first part briefly defines quaternions and includes sources for more information. Then, the module and ring structures of the n-dimensional quaternionic space will be presented, and the module structure of the n-dimensional quaternionic space over the n-dimensional real space will be reviewed, [11]. Finally, information about the Frenet-Serret formulas is given.

2.1. REAL QUATERNION SPACE

The ring of real quaternions was discovered by W. R. Hamilton in 1843, [13]. The set of all real quaternions can be represented as

$$\mathbb{H} = \{q \;\mid\; q = a\vec{e}_0 + b\vec{e}_1 + c\vec{e}_2 + d\vec{e}_3,\ a, b, c, d \in \mathbb{R},\ \vec{e}_0, \vec{e}_1, \vec{e}_2, \vec{e}_3 \in \mathbb{R}^4\}.$$

This set is a vector space over $\mathbb{R}$ with the basis $\{1, i, j, k\}$, satisfying the equation $i^2 = j^2 = k^2 = ijk = -1$.

2.2. n-DIMENSIONAL QUATERNIONIC SPACE

The set of all real quaternions n-vectors $\mathbb{H}^n$ can be represented as,

$$\mathbb{H}^n = \mathbb{H} \times \mathbb{H} \times \ldots \times \mathbb{H} = \{\vec{q} = (q_i) = (q_1, q_2, \ldots, q_n) \mid q_i \in \mathbb{H}, 1 \leq i \leq n\}.$$

Each component of this set are quaternions $q_i = S_{q_i} + V_{q_i} \in \mathbb{H}$, hence the components can be written as,

$$\vec{q} = \left(S_{q_1}, S_{q_2}, \ldots, S_{q_n}\right) + \left(V_{q_1}, V_{q_2}, \ldots, V_{q_n}\right) = S_{\vec{q}} + V_{\vec{q}}.$$

That is, each quaternion n-vector can be written as the sum of the *real part* $S_{\vec{q}}$ and the *vector part* $V_{\vec{q}}$. As a matter of course, the set of *spatial quaternions n-vectors* are defined as,

$$\mathbb{H}_P^n = \mathbb{H}_P \times \mathbb{H}_P \times \ldots \times \mathbb{H}_P = \{\vec{q} = (q_i) = (q_1, q_2, \ldots, q_n) \mid q_i \in \mathbb{H}_P, 1 \leq i \leq n\}.$$

Addition operation $\boxplus : \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{H}^n$ of two quaternions n-vectors is defined as,

$$\vec{q} \boxplus \vec{p} = (q_1 \oplus p_1, q_2 \oplus p_2, \ldots, q_n \oplus p_n) \tag{4}$$

and scalar multiplication $\odot : \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{H}^n$ of the quaternion n-vector and the scalar quaternion is defined as,

$$\vec{q} \odot \lambda = (q_i) \odot \lambda = (q_i \times \lambda) = (q_1 \times \lambda, q_2 \times \lambda, \ldots, q_n \times \lambda) \tag{5}$$

for all $\vec{q} = (q_i) = (q_1, q_2, \ldots, q_n), \vec{p} = (p_i) = (p_1, p_2, \ldots, p_n) \in \mathbb{H}^n$, $\lambda \in \mathbb{H}$.

Corollary 1. $\{\mathbb{H}^n, \boxplus, \mathbb{H}, \oplus, \times, \odot\} = \mathbb{H}^n$ is a right $\mathbb{H}$-module with $\dim \mathbb{H}^n = n$.

Inner product function $\langle , \rangle_{\mathbb{H}^n} : \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{H}^n$ for the quaternion n-vectors is defined as,

$$\langle \vec{q}, \vec{p} \rangle_{\mathbb{H}^n} = \sum_{i=1}^{n} \bar{q}_i \times p_i \tag{6}$$

for all $\vec{q} = (q_i), \vec{p} = (p_i) \in \mathbb{H}^n, q_i, p_i \in \mathbb{H}$ $(1 \leq i \leq n)$.

Norm function $\|.\|_{\mathbb{H}^n} : \mathbb{H}^n \to \mathbb{R}$ is defined as,

$$\begin{aligned} \|\vec{q}\|_{\mathbb{H}^n} = \sqrt{\langle \vec{q}, \vec{q} \rangle_{\mathbb{H}^n}} = \sqrt{\sum_{i=1}^{n} \bar{q}_i \times q_i} &= \sqrt{\sum_{i=1}^{n} \|q_i\|^2} \\ &= \sqrt{\|q_1\|^2 + \|q_2\|^2 + \cdots + \|q_n\|^2} \end{aligned} \tag{7}$$

for all $\vec{q} = (q_i) \in \mathbb{H}^n$.

Multiplication operation $\boxtimes : \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{H}^n$ of two quaternions n-vectors is defined as,

$$\vec{q} \boxtimes \vec{p} = (q_1 \times p_1, q_2 \times p_2, \ldots, q_n \times p_n) \tag{8}$$

for all $\vec{q}=(q_i)$, $\vec{p}=(p_i)\in\mathbb{H}^n$, $q_i,p_i\in\mathbb{H}$ $(1\leq i\leq n)$. This operation is called *quaternion n-vectors componentwise multiplication* on $\mathbb{H}^n$. Here, the operation "×" is multiplication between two quaternions on $\mathbb{H}$.

Corollary 2. $(\mathbb{H}^n,\boxplus,\boxtimes)$ is a unitary ring.

The conjugate of a quaternion n-vector is defined as,

$$\bar{\vec{q}}=(\bar{q}_1,\bar{q}_2,\dots,\bar{q}_n)=\left(S_{q_1}-V_{q_1},S_{q_2}-V_{q_2},\dots,S_{q_n}-V_{q_n}\right)=S_{\vec{q}}-V_{\vec{q}}$$

for all $\vec{q}=(q_i)\in\mathbb{H}^n$.

2.3. MODULE STRUCTURE OF n-DIMENSIONAL QUATERNIONIC SPACE OVER n-DIMENSIONAL REAL SPACE

In this part, we first define a scalar multiplication operation that will allow us to construct the module structure. Then, the inner product and norm functions of this module will be characterized.

Multiplication $\boxdot:\mathbb{H}^n\times\mathbb{R}^n\to\mathbb{H}^n$ of quaternion n-vector with real n-vector is defined by using (1) as,

$$\begin{aligned}\vec{q}\boxdot\vec{\lambda}&=\left(\vec{A}\vec{e}_0+\vec{B}\vec{e}_1+\vec{C}\vec{e}_2+\vec{D}\vec{e}_3\right)\boxdot\vec{\lambda}\\&=\left(\vec{A}\boxtimes_{\mathbb{R}^n}\vec{\lambda}\right)\vec{e}_0+\left(\vec{B}\boxtimes_{\mathbb{R}^n}\vec{\lambda}\right)\vec{e}_1+\left(\vec{C}\boxtimes_{\mathbb{R}^n}\vec{\lambda}\right)\vec{e}_2+\left(\vec{D}\boxtimes_{\mathbb{R}^n}\vec{\lambda}\right)\vec{e}_3\end{aligned}$$

for all $\vec{q}=\vec{A}\vec{e}_0+\vec{B}\vec{e}_1+\vec{C}\vec{e}_2+\vec{D}\vec{e}_3\in\mathbb{H}^n$ and $\vec{\lambda}\in\mathbb{R}^n$.

Multiplication operation is also defined using (5) as

$$\vec{q}\boxdot\vec{\lambda}=(q_1\odot\lambda_1,q_2\odot\lambda_2,\dots,q_n\odot\lambda_n)$$

for all $\vec{q}=(q_i)\in\mathbb{H}^n, q_i\in\mathbb{H}$, and $\vec{\lambda}=(\lambda_i)\in\mathbb{R}^n(1\leq i\leq n)$.

Corollary 3. $\{\mathbb{H}^n,\boxplus,\mathbb{R}^n,+,\boxtimes_{\mathbb{R}^n},\boxdot\}$ is a module with $\dim\mathbb{H}^n=4$, which means that $\mathbb{H}^n$ is 4-dimensional over the space of real n-vectors. And the space of the quaternion n-vectors can be denoted by $\mathbb{H}^n=Sp\{\vec{e}_0,\vec{e}_1,\vec{e}_2,\vec{e}_3\}$.

Inner product from $\mathbb{H}^n\times\mathbb{H}^n$ to $\mathbb{R}^n$ is defined as,

$$\langle\vec{q},\vec{p}\rangle_{\mathbb{H}^n}^{\mathbb{R}^n}=\frac{1}{2}\left[\left(\vec{q}\boxtimes\bar{\vec{p}}\right)\boxplus\left(\vec{p}\boxtimes\bar{\vec{q}}\right)\right] \tag{9}$$

for all $\vec{q},\vec{p}\in\mathbb{H}^n$.

Norm function from $\mathbb{H}^n$ to $\mathbb{R}^n$ is defined as,

$$\|\vec{q}\|_{\mathbb{H}^n}^{\mathbb{R}^n}=\sqrt{\langle\vec{q},\vec{q}\rangle_{\mathbb{H}^n}^{\mathbb{R}^n}}=\sqrt{\vec{q}\boxtimes\bar{\vec{q}}}=(\|q_1\|,\|q_2\|,\dots,\|q_n\|) \tag{10}$$

for all $\vec{q},\vec{p}\in\mathbb{H}^n$. Here, the norm in the components of the function is the norm operation given in (7).

2.4. FRENET-SERRET FRAME

This section presents Frenet-Serret formulations for the quaternionic curves examined by [14]. Throughout the paper, we assume that all curves have an arc-length parameter.

Theorem 4. A spatial quaternionic curve $\gamma: I = [0,1] \subset \mathbb{R} \to \mathbb{H}_P$ parameterized by $s \in I \subset \mathbb{R}$ arc-length. With $\{t(s), n(s), b(s)\}$ being the Frenet frame field at the point $\gamma(s)$, the Frenet-Serret Formulae of a spatial quaternionic curve are given by,

$$t'(s) = \kappa n(s)$$

$$n'(s) = -\kappa t(s) + \tau b(s)$$

$$b'(s) = -\tau n(s).$$

Here, κ and τ are the principal curvature and torsion of γ, respectively.

Theorem 5. Let $\beta: I \subset \mathbb{R} \to \mathbb{H}$ be a unit speed quaternionic curve given in $\mathbb{H}$. Suppose that $T = \beta'$ is the tangent vector of β. Now, if N, B, and D vectors are selected as $N = t \times T$, $B = n \times T$, and $D = b \times T$, then the vector system $\{T, N, B, D\}$ is obtained with the help of the $\{t, n, b\}$ frame. The Frenet-Serret Formulae of the quaternionic curve at the point β are given by,

$$T'(s) = K(s)N(s)$$

$$N'(s) = -K(s)T(s) + \kappa(s)B(s)$$

$$B'(s) = -\kappa(s)N(s) + \big(\tau(s) - K(s)\big)D(s)$$

$$D'(s) = -\big(\tau(s) - K(s)\big)B(s).$$

Here, K is the curvature of β, [14].

3. FRENET-SERRET FORMULAS OF n-DIMENSIONAL QUATERNIONIC SPACE

While standard quaternionic Frenet-Serret formulas describe the motion of single rigid bodies, this generalization to the $\mathbb{H}^n$ space allows for the formulation of synchronized trajectory analyses of interacting multi-agent robotic systems or multi-joint mechanisms under a single algebraic framework.

In this section, curves will be given more extensively on the quaternion n-vectors space $\mathbb{H}^n$. First, the spatial quaternion n-vectors space $\mathbb{H}_P^n$ will be studied and generalized. Then, the Frenet-Serret formulas for these curves will be obtained, and the results will be derived by considering the componentwise metric defined in (3).

The length of an n-vector in the n-dimensional quaternionic space will be determined by considering the componentwise metric. In addition, proofs will be made for quaternion n-vectors with the help of the previously defined inner product and norm operations, the concept of orthogonality, conjugate operation properties, and derivative properties.

The n-vector systems $\{\vec{t},\ \vec{n},\ \vec{b}\}$ and $\{\vec{T},\ \vec{N},\ \vec{B},\ \vec{D}\}$ were obtained using [14].

3.1. SPATIAL QUATERNION CURVES IN $\mathbb{H}_P^n$

Definition 1. Let

$$\mathbb{H}_P^n = \{\vec{\gamma} = (\gamma_1, \gamma_2, \dots, \gamma_n) \ \mid\ \gamma_i + \bar{\gamma}_i = 0,\ \gamma_i = \gamma_{i1}\vec{e}_1 + \gamma_{i2}\vec{e}_2 + \gamma_{i3}\vec{e}_3 \in \mathbb{H}_P\}$$

be the spatial quaternion n-vectors space, where $1 \leq\ i \leq n$. In this space, the function,

$$\vec{\gamma}: I \subset \mathbb{R} \to \mathbb{H}_P^n$$
$$s \to \vec{\gamma}(s) = \big(\gamma_1(s), \gamma_2(s), \dots, \gamma_n(s)\big)$$
$$= \left(\sum_{j=1}^{3} \gamma_{1j}\,(s)\vec{e}_j, \sum_{j=1}^{3} \gamma_{2j}\,(s)\vec{e}_j, \dots, \sum_{j=1}^{3} \gamma_{nj}\,(s)\vec{e}_j\right) = \left(\sum_{j=1}^{3} \gamma_{ij}\,(s)\vec{e}_j\right)_{i=1}^{n}$$

is called *n-dimensional spatial quaternionic curve* for $j = 1,2,3$ where $s\ \in I$ is an arc-length parameter. Here, each function γ_i $(1 \leq i \leq n)$ are spatial quaternionic curves in $\mathbb{H}_P$.

Definition 2. Let $\vec{\gamma}: I \subset \mathbb{R} \to \mathbb{H}_P^n$ be n-dimensional spatial quaternionic curve with $I \subset \mathbb{R}$ unit interval and arc-length parameter $s \in I$ in $\mathbb{H}_P^n$. Then, $\vec{\gamma}$ is called *n-dimensional unit speed spatial quaternionic curve*, if $\|\vec{\gamma}'(s)\|_{\mathbb{H}_P^n}^{\mathbb{R}^n} = \vec{1}$ where, $\vec{\gamma}'(s)$ is the n-vector at each point of $\vec{\gamma}$. In this case, by considering the equation $\vec{\gamma}'(s) \boxtimes \overline{\vec{\gamma}'}(s) = \vec{1}$ for unit speed curves,

$$\left[\|\vec{\gamma}'(s)\|_{\mathbb{H}_P^n}^{\mathbb{R}^n}\right]^2 = \langle \vec{\gamma}'(s), \vec{\gamma}'(s) \rangle_{\mathbb{H}_P^n}^{\mathbb{R}^n} = \frac{1}{2}\left[\left(\vec{\gamma}'(s) \boxtimes \overline{\vec{\gamma}'}(s)\right) \boxplus \left(\vec{\gamma}'(s) \boxtimes \overline{\vec{\gamma}'}(s)\right)\right] = \vec{1}$$

is obtained.

Definition 3. If $\|\vec{\gamma}'(s)\|_{\mathbb{H}_P^n}^{\mathbb{R}^n} \neq \vec{0}$ or $\overrightarrow{\gamma'}(s) \neq \vec{0}$ for each real parameter s and $\vec{0} = (0,0, \dots ,0)$, then $\vec{\gamma}$ is called *n-dimensional spatial quaternionic regular curve*.

Definition 4. Let $\vec{\gamma}: I \subset \mathbb{R} \to \mathbb{H}_P^n$ be n-dimensional unit speed spatial quaternionic curve given in $\mathbb{H}_P^n$. The n-vector $\vec{t}(s) = \vec{\gamma}'(s) = \left(\sum_{j=1}^{3} \gamma_{ij}'(s)\vec{e}_j\right)_{i=1}^{n}$ is called the unit tangent n-vector of $\vec{\gamma}$.

Theorem 6. Let $\vec{\gamma}: I \subset \mathbb{R} \to \mathbb{H}_P^n$ be n-dimensional unit speed spatial quaternionic curve given in $\mathbb{H}_P^n$. Then, $\vec{t}$ and $\vec{t}'$ are orthogonal, and also, $\vec{t}' \boxtimes \overline{\vec{t}}$ is a spatial quaternion n-vector.

Proof: By taking the derivative of both sides of

$$\left[\|\vec{\gamma}'(s)\|_{\mathbb{H}_P^n}^{\mathbb{R}^n}\right]^2 = \left[\left\|\vec{t}(s)\right\|_{\mathbb{H}_P^n}^{\mathbb{R}^n}\right]^2 = \vec{t}(s) \boxtimes \overline{\vec{t}}(s) = \vec{1},$$

then the equation,

$$\left(\vec{t}' \boxtimes \overline{\vec{t}}\right) \boxplus \left(\vec{t} \boxtimes \left(\overline{\vec{t}}\right)'\right) = \vec{0} \tag{11}$$

is obtained. Since $\vec{t}(s)=\left(\sum_{j=1}^{3}\gamma'_{ij}(s)\vec{e}_j\right)_{i=1}^{n}$ is a spatial quaternion n-vector, $\overline{\vec{t}}(s)=-\left(\sum_{j=1}^{3}\gamma'_{ij}(s)\vec{e}_j\right)_{i=1}^{n}$ can be written in accordance with the properties of the conjugate operation. By using the equations $\overline{\vec{t}'}(s)=-\left(\sum_{j=1}^{3}\gamma''_{ij}(s)\vec{e}_j\right)_{i=1}^{n}$ and $\left(\overline{\vec{t}}\right)'(s)=-\left(\sum_{j=1}^{3}\gamma''_{ij}(s)\vec{e}_j\right)_{i=1}^{n}$,

$$\overline{\vec{t}'}(s)=\left(\overline{\vec{t}}\right)'(s) \tag{12}$$

is obtained. Thus, with the help of (11) and (12),

$$\langle\vec{t},\vec{t}'\rangle_{\mathbb{H}_P^n}^{\mathbb{R}^n}=\frac{1}{2}\left[\left(\vec{t}'\boxtimes\overline{\vec{t}}\right)\boxplus\left(\vec{t}\boxtimes\overline{\vec{t}'}\right)\right]=\vec{0}.$$

That is, $\vec{t}$ and $\vec{t}'$ are orthogonal in $\mathbb{H}^n$. It can be easily seen that $\vec{t}'\boxtimes\overline{\vec{t}}$ is a spatial quaternion n-vector from the expression,

$$\left(\vec{t}'\boxtimes\overline{\vec{t}}\right)\boxplus\left(\vec{t}\boxtimes\left(\overline{\vec{t}}\right)'\right)=\left(\vec{t}'\boxtimes\overline{\vec{t}}\right)\boxplus\left(\overline{\overline{\vec{t}}}\boxtimes\overline{\vec{t}'}\right)=\left(\vec{t}'\boxtimes\overline{\vec{t}}\right)\boxplus\left(\overline{\vec{t}'\boxtimes\overline{\vec{t}}}\right)=\vec{0}.$$

Definition 5. Let $\vec{\gamma}: I\subset\mathbb{R}\to\mathbb{H}_P^n$ be n-dimensional unit speed spatial quaternionic curve given in $\mathbb{H}_P^n$. The following n-vector,

$$\vec{n}(s)=\vec{t}'(s)\boxdot_P\left[\left\|\vec{t}'(s)\right\|_{\mathbb{H}_P^n}^{\mathbb{R}^n}\right]^{-1} \tag{13}$$

is a unit n-vector and since $\vec{t}$ and $\overrightarrow{t'}$ are orthogonal, it is clear that $\left\langle\vec{t}(s),\vec{n}(s)\right\rangle_{\mathbb{H}_P^n}^{\mathbb{R}^n}=\vec{0}$ from the choice made in (13).

Lemma 7. Selecting $\vec{b}(s)=\vec{t}(s)\boxtimes\vec{n}(s)$, now provides the following features:

(i) $\vec{t}(s)\boxtimes\vec{n}(s)=\vec{b}(s)=-\vec{n}(s)\boxtimes\vec{t}(s)$

(ii) $\vec{t}(s)\boxtimes\vec{b}(s)=-\vec{n}(s)=-\vec{b}(s)\boxtimes\vec{t}(s)$

(iii) $\vec{n}(s)\boxtimes\vec{b}(s)=\vec{t}(s)=-\vec{b}(s)\boxtimes\vec{n}(s)$

(iv) $\vec{t}\boxtimes\overline{\vec{n}}=\overline{\vec{b}}$

(v) $\vec{n}\boxtimes\overline{\vec{t}}=\vec{b}$

(vi) $\vec{t}\boxtimes\overline{\vec{b}}=\vec{n}$ and $\vec{b}\boxtimes\overline{\vec{t}}=\overline{\vec{n}}$

(vii) $\vec{n}\boxtimes\overline{\vec{b}}=\overline{\vec{t}}$ and $\vec{b}\boxtimes\overline{\vec{n}}=\vec{t}$

With the help of these features, the following equalities are obtained.

$$\left\langle\vec{t}(s),\vec{n}(s)\right\rangle_{\mathbb{H}_P^n}^{\mathbb{R}^n}=\frac{1}{2}\left[\left(\vec{t}\boxtimes\overline{\vec{n}}\right)\boxplus\left(\vec{n}\boxtimes\overline{\vec{t}}\right)\right]=\frac{1}{2}\left[\overline{\vec{b}}\boxplus\vec{b}\right]=\vec{0} \tag{14}$$

$$\left\langle\vec{t}(s),\vec{b}(s)\right\rangle_{\mathbb{H}_P^n}^{\mathbb{R}^n}=\frac{1}{2}\left[\left(\vec{t}\boxtimes\overline{\vec{b}}\right)\boxplus\left(b\boxtimes\overline{\vec{t}}\right)\right]=\frac{1}{2}\left[\vec{n}\boxplus\overline{\vec{n}}\right]=\vec{0} \tag{15}$$

$$\left\langle\vec{n}(s),\vec{b}(s)\right\rangle_{\mathbb{H}_P^n}^{\mathbb{R}^n}=\frac{1}{2}\left[\left(\vec{n}\boxtimes\overline{\vec{b}}\right)\boxplus\left(\vec{b}\boxtimes\overline{\vec{n}}\right)\right]=\frac{1}{2}\left[\overline{\vec{t}}\boxplus\vec{t}\right]=\vec{0} \tag{16}$$

$$\left\langle \vec{b}(s), \vec{b}(s) \right\rangle_{\mathbb{H}_P^n}^{\mathbb{R}^n} = \frac{1}{2}\left[\left(\vec{b} \boxtimes \overline{\vec{b}}\right) \boxplus \left(\vec{b} \boxtimes \overline{\vec{b}}\right)\right] = \vec{b} \boxtimes \overline{\vec{b}} = -\left\langle \vec{b}, \overline{\vec{b}} \right\rangle_{\mathbb{H}_P^n}^{\mathbb{R}^n} \boxplus \vec{b} \wedge_{\mathbb{H}_P^n} \overline{\vec{b}} = \vec{1}. \tag{17}$$

So, the orthonormal n-vector system $\{\vec{t}(s), \vec{n}(s), \vec{b}(s)\}$ is acquired using the componentwise metric. The n-vectors $\vec{t}(s), \vec{n}(s)$, and $\vec{b}(s)$ are also called *Frenet n-vectors of the unit speed curve* in $\mathbb{H}_P^n$.

Definition 6. Curvature of $\vec{\gamma}$ on $\mathbb{H}_P^n$ is defined as,

$$\vec{\kappa}(s) = \left\|\vec{t}'(s)\right\|_{\mathbb{H}_P^n}^{\mathbb{R}^n}. \tag{18}$$

Theorem 8. Let $\vec{t}$ be the unit tangent n-vector of the unit speed curve $\vec{\gamma}$ and $\vec{\kappa}$ be the curvature on $\mathbb{H}_P^n$. Then the following equality is acquired,

$$\vec{t}'(s) = \vec{n}(s) \boxdot_P \vec{\kappa}(s). \tag{19}$$

Theorem 9. Let for every $s \in I$ of $\vec{\gamma}$, $\{\vec{t}(s), \vec{n}(s), \vec{b}(s)\}$ be the system at the point $\vec{\gamma}(s)$, and $\vec{\kappa}(s)$ and $\vec{\tau}(s)$ be its curvatures. The relationship between the derivatives $\overrightarrow{t'}(s)$, $\vec{n}'(s)$, $\vec{b}'(s)$ of the n-vectors $\vec{t}(s)$, $\vec{n}(s)$, $\vec{b}(s)$ and the curvatures are given by,

$$\begin{aligned}
\vec{t}'(s) &= \vec{n}(s) \boxdot_P \vec{\kappa}(s) \\
\vec{n}'(s) &= \left(\vec{t}(s) \boxdot_P (-\vec{\kappa}(s))\right) \boxplus \left(\vec{b}(s) \boxdot_P \vec{\tau}(s)\right) \\
\vec{b}'(s) &= \vec{n}(s) \boxdot_P (-\vec{\tau}(s)).
\end{aligned}$$

Proof: Since, $\overrightarrow{t'}(s)$, $\vec{n}'(s)$, $\vec{b}'(s)$ forms an orthonormal system, the following equations can be written:

$$\vec{t}'(s) = \left(\vec{t}(s) \boxdot_P a_{11}\right) \boxplus \left(\vec{n}(s) \boxdot_P a_{12}\right) \boxplus \left(\vec{b}(s) \boxdot_P a_{13}\right) \tag{20}$$

$$\vec{n}'(s) = \left(\vec{t}(s) \boxdot_P a_{21}\right) \boxplus \left(\vec{n}(s) \boxdot_P a_{22}\right) \boxplus \left(\vec{b}(s) \boxdot_P a_{23}\right) \tag{21}$$

$$\vec{b}'(s) = \left(\vec{t}(s) \boxdot_P a_{31}\right) \boxplus \left(\vec{n}(s) \boxdot_P a_{32}\right) \boxplus \left(\vec{b}(s) \boxdot_P a_{33}\right). \tag{22}$$

From these equations, the coefficients $a_{ij} \in \mathbb{R}^n$ $(1 \leq i, j \leq 3)$ can be evaluated.

If the expression (20) is inner producted by $\vec{t}, \vec{n}$, and $\vec{b}$ respectively, then with the help of the equations (14), (15), and Theorem 6,

$$\begin{aligned}
\left\langle \langle \vec{t}', \vec{t} \rangle_{\mathbb{H}_P^n}^{\mathbb{R}^n} = \left(\langle \vec{t}, \vec{t} \rangle_{\mathbb{H}_P^n}^{\mathbb{R}^n} \boxdot_P a_{11}\right) \boxplus \left(\langle \vec{n}, \vec{t} \rangle_{\mathbb{H}_P^n}^{\mathbb{R}^n} \boxdot_P a_{12}\right) \boxplus \left(\langle \vec{b}, \vec{t} \rangle_{\mathbb{H}_P^n}^{\mathbb{R}^n} \boxdot_P a_{13}\right)\right\rangle \\
= a_{11} = \vec{0}
\end{aligned}$$

can be written. Similarly,

$$\begin{aligned}
\langle \vec{t}', \vec{n} \rangle_{\mathbb{H}_P^n}^{\mathbb{R}^n} = \left(\langle \vec{t}, \vec{n} \rangle_{\mathbb{H}_P^n}^{\mathbb{R}^n} \boxdot_P a_{11}\right) \boxplus \left(\langle \vec{n}, \vec{n} \rangle_{\mathbb{H}_P^n}^{\mathbb{R}^n} \boxdot_P a_{12}\right) \boxplus \left(\langle \vec{b}, \vec{n} \rangle_{\mathbb{H}_P^n}^{\mathbb{R}^n} \boxdot_P a_{13}\right) \\
= a_{12} = \vec{0}
\end{aligned}$$

and by using (14), (16), and (19)

$$\langle \vec{t}', \vec{n} \rangle_{\mathbb{H}_P^n}^{\mathbb{R}^n} = \langle \vec{n} \boxdot_P \vec{\kappa}, \vec{n} \rangle_{\mathbb{H}_P^n}^{\mathbb{R}^n} = \langle \vec{n}, \vec{n} \rangle_{\mathbb{H}_P^n}^{\mathbb{R}^n} \boxtimes_{\mathbb{R}^n} \vec{\kappa} = \vec{\kappa} = a_{12}$$

is obtained. Also,

$$\langle \vec{t}', \vec{b} \rangle_{\mathbb{H}_P^n}^{\mathbb{R}^n} = \left(\langle \vec{t}, \vec{b} \rangle_{\mathbb{H}_P^n}^{\mathbb{R}^n} \boxdot_P a_{11} \right) \boxplus \left(\langle \vec{n}, \vec{b} \rangle_{\mathbb{H}_P^n}^{\mathbb{R}^n} \boxdot_P a_{12} \right) \boxplus \left(\langle \vec{b}, \vec{b} \rangle_{\mathbb{H}_P^n}^{\mathbb{R}^n} \boxdot_P a_{13} \right) = a_{13}$$

can be written, then, by using (15), (16), and (19)

$$\langle \vec{t}', \vec{b} \rangle_{\mathbb{H}_P^n}^{\mathbb{R}^n} = \langle \vec{n} \boxdot_P \vec{\kappa}, \vec{b} \rangle_{\mathbb{H}_P^n}^{\mathbb{R}^n} = \langle \vec{n}, \vec{b} \rangle_{\mathbb{H}_P^n}^{\mathbb{R}^n} \boxtimes_{\mathbb{R}^n} \vec{\kappa} = \vec{0} = a_{13}$$

is found. Finally, the expression (20) is obtained as,

$$\vec{t}'(s) = \vec{n}(s) \boxdot_P \vec{\kappa}(s). \tag{23}$$

If the expression (21) is inner producted by $\vec{t}$, $\vec{n}$, and $\vec{b}$, respectively,

$$\left\langle \langle \vec{n}', \vec{t} \rangle_{\mathbb{H}_P^n}^{\mathbb{R}^n} = \left(\langle \vec{t}, \vec{t} \rangle_{\mathbb{H}_P^n}^{\mathbb{R}^n} \boxdot_P a_{21} \right) \boxplus \left(\langle \vec{n}, \vec{t} \rangle_{\mathbb{H}_P^n}^{\mathbb{R}^n} \boxdot_P a_{22} \right) \boxplus \left(\langle \vec{b}, \vec{t} \rangle_{\mathbb{H}_P^n}^{\mathbb{R}^n} \boxdot_P a_{23} \right) \right\rangle = a_{21}$$

can be written. By differentiating equation (14),

$$\frac{1}{2}\left[\left(\vec{t}' \boxtimes \overline{\vec{n}} \right) \boxplus \left(\vec{t} \boxtimes \left(\overline{\vec{n}} \right)' \right) \boxplus \left(\vec{n}' \boxtimes \overline{\vec{t}} \right) \boxplus \left(\vec{n} \boxtimes \left(\overline{\vec{t}} \right)' \right) \right] = \vec{0}$$

is found. Since $\vec{t}$ and $\vec{n}$ are spatial quaternion n-vectors, and equation (12) is valid for $\vec{t}$ and $\vec{n}$, if the equality is rearranged,

$$\frac{1}{2}\left[\left(\vec{t}' \boxtimes \overline{\vec{n}} \right) \boxplus \left(\vec{n} \boxtimes \left(\overline{\vec{t}'} \right) \right) \right] \boxplus \frac{1}{2}\left[\left(\vec{t} \boxtimes \left(\overline{\vec{n}'} \right) \right) \boxplus \left(\vec{n}' \boxtimes \overline{\vec{t}} \right) \right] = \vec{0}$$

is acquired, and this equation provides $\langle \vec{t}', \vec{n} \rangle_{\mathbb{H}_P^n}^{\mathbb{R}^n} + \langle \vec{t}, \vec{n}' \rangle_{\mathbb{H}_P^n}^{\mathbb{R}^n} = \vec{0}$. Here, it yields to $\langle \vec{t}, \vec{n}' \rangle_{\mathbb{H}_P^n}^{\mathbb{R}^n} = -\langle \vec{t}', \vec{n} \rangle_{\mathbb{H}_P^n}^{\mathbb{R}^n} = -a_{12} = -\vec{\kappa}$. Also,

$$\langle \vec{n}', \vec{n} \rangle_{\mathbb{H}_P^n}^{\mathbb{R}^n} = \left(\langle \vec{t}, \vec{n} \rangle_{\mathbb{H}_P^n}^{\mathbb{R}^n} \boxdot_P a_{21} \right) \boxplus \left(\langle \vec{n}, \vec{n} \rangle_{\mathbb{H}_P^n}^{\mathbb{R}^n} \boxdot_P a_{22} \right) \boxplus \left(\langle \vec{b}, \vec{n} \rangle_{\mathbb{H}_P^n}^{\mathbb{R}^n} \boxdot_P a_{23} \right) = a_{22} = \vec{0}$$

can be found. Furthermore, the equation,

$$\langle \vec{n}', b \rangle_{\mathbb{H}_P^n}^{\mathbb{R}^n} = \left(\langle \vec{t}, \vec{b} \rangle_{\mathbb{H}_P^n}^{\mathbb{R}^n} \boxdot_P a_{21} \right) \boxplus \left(\langle \vec{n}, \vec{b} \rangle_{\mathbb{H}_P^n}^{\mathbb{R}^n} \boxdot_P a_{22} \right) \boxplus \left(\langle \vec{b}, \vec{b} \rangle_{\mathbb{H}_P^n}^{\mathbb{R}^n} \boxdot_P a_{23} \right) = a_{23}$$

can be written. Now, by differentiating equation (16),

$$\frac{1}{2}\left[\left(\vec{n}' \boxtimes \overline{\vec{b}} \right) \boxplus \left(\vec{n} \boxtimes \left(\overline{\vec{b}} \right)' \right) \boxplus \left(\vec{b}' \boxtimes \overline{\vec{n}} \right) \boxplus \left(\vec{b} \boxtimes \left(\overline{\vec{n}} \right)' \right) \right] = \vec{0}$$

is obtained. Since $\vec{n}$ and $\vec{b}$ are spatial quaternion n-vectors and equality (12) is valid for $\vec{n}$ and $\vec{b}$, if the previous equality is rearranged,

$$\frac{1}{2}\left[\left(\vec{n}' \boxtimes \overline{\vec{b}}\right) \boxplus \left(\vec{b} \boxtimes \left(\overline{\vec{n}'}\right)\right)\right] \boxplus \frac{1}{2}\left[\left(\vec{n} \boxtimes \left(\overline{\vec{b}'}\right)\right) \boxplus \left(\vec{b}' \boxtimes \overline{\vec{n}}\right)\right] = \vec{0}$$

is acquired and this provides the equation, $\left\langle \vec{n}', \vec{b} \right\rangle_{\mathbb{H}_P^n}^{\mathbb{R}^n} + \left\langle \vec{n}, \vec{b}' \right\rangle_{\mathbb{H}_P^n}^{\mathbb{R}^n} = \vec{0}$. Here, since $\left\langle \vec{n}', \vec{b} \right\rangle_{\mathbb{H}_P^n}^{\mathbb{R}^n} = a_{23} = -\left\langle \vec{n}, \vec{b}' \right\rangle_{\mathbb{H}_P^n}^{\mathbb{R}^n}$ and $a_{23} = \vec{\tau}(s)$ is taken, the expression (21), becomes,

$$\vec{n}'(s) = \left(\vec{t}(s) \boxdot_P - \vec{\kappa}(s)\right) \boxplus \left(\vec{b}(s) \boxdot_P \vec{\tau}(s)\right). \tag{24}$$

The resulting expression,

$$a_{23} = \left\langle \vec{n}', \vec{b} \right\rangle_{\mathbb{H}_P^n}^{\mathbb{R}^n} = \vec{\tau}(s) \tag{25}$$

is called *second curvature* or *torsion* of the unit speed curve $\vec{\gamma}$ in $\mathbb{H}_P^n$.

If the expression (22) is inner producted by $\vec{t}$, $\vec{n}$, and $\vec{b}$, respectively,

$$\langle \vec{b}', \vec{t} \rangle_{\mathbb{H}_P^n}^{\mathbb{R}^n} = \left(\left\langle \vec{t}, \vec{t} \right\rangle_{\mathbb{H}_P^n}^{\mathbb{R}^n} \boxdot_P a_{31}\right) \boxplus \left(\left\langle \vec{n}, \vec{t} \right\rangle_{\mathbb{H}_P^n}^{\mathbb{R}^n} \boxdot_P a_{32}\right) \boxplus \left(\left\langle \vec{b}, \vec{t} \right\rangle_{\mathbb{H}_P^n}^{\mathbb{R}^n} \boxdot_P a_{33}\right) = a_{31} = \vec{0}$$

$$\langle \vec{b}', \vec{n} \rangle_{\mathbb{H}_P^n}^{\mathbb{R}^n} = \left(\left\langle \vec{t}, \vec{n} \right\rangle_{\mathbb{H}_P^n}^{\mathbb{R}^n} \boxdot_P a_{31}\right) \boxplus \left(\langle \vec{n}, \vec{n} \rangle_{\mathbb{H}_P^n}^{\mathbb{R}^n} \boxdot_P a_{32}\right) \boxplus \left(\left\langle \vec{b}, \vec{n} \right\rangle_{\mathbb{H}_P^n}^{\mathbb{R}^n} \boxdot_P a_{33}\right) = a_{32} = -\vec{\tau}(s)$$

$$\langle \vec{b}', \vec{b} \rangle_{\mathbb{H}_P^n}^{\mathbb{R}^n} = \left(\left\langle \vec{t}, \vec{b} \right\rangle_{\mathbb{H}_P^n}^{\mathbb{R}^n} \boxdot_P a_{31}\right) \boxplus \left(\left\langle \vec{n}, \vec{b} \right\rangle_{\mathbb{H}_P^n}^{\mathbb{R}^n} \boxdot_P a_{32}\right) \boxplus \left(\left\langle \vec{b}, \vec{b} \right\rangle_{\mathbb{H}_P^n}^{\mathbb{R}^n} \boxdot_P a_{33}\right) = a_{33} = \vec{0}$$

can be written in the same manner. Thus, the expression (22) becomes,

$$\vec{b}'(s) = \vec{n}(s) \boxdot_P (-\vec{\tau}(s)). \tag{26}$$

Corollary 10. The curvature of $\vec{\gamma}$ given by the equality (18) can also be written as,

$$\vec{\kappa}(s) = (\kappa_1(s), \kappa_2(s), \dots, \kappa_n(s))$$

in $\mathbb{H}_P^n$. Here, each value of $\kappa_i(s)$ $(1 \leq i \leq n)$ is the curvature of quaternionic curves γ_i in $\mathbb{H}$.

Corollary 11. The torsion of $\vec{\gamma}$ given by (25) can also be written as,

$$\vec{\tau}(s) = (\tau_1(s), \tau_2(s), \dots, \tau_n(s))$$

in $\mathbb{H}_P^n$. Here, each value of $\tau_i(s)$ $(1 \leq i \leq n)$ is the torsion of quaternionic curves γ_i in $\mathbb{H}$.

Example 1. Let the curve $\vec{\gamma}: I \subset \mathbb{R} \to \mathbb{H}_P^2$ be given as,

$$\vec{\gamma}(s) = \left(\left(\frac{s}{2\sqrt{2}} - \frac{\sqrt{3}}{2\sqrt{2}}\sin s\right)\vec{e}_1 + \left(\frac{s}{2\sqrt{2}} + \frac{\sqrt{3}}{2\sqrt{2}}\sin s\right)\vec{e}_2 - \frac{\sqrt{3}}{2}\cos s\,\vec{e}_3, \right.$$
$$\left. \left(\frac{1}{3}\cos s - \frac{\sqrt{7}}{3}\sin s\right)\vec{e}_1 + \frac{s}{3}\vec{e}_2 + \left(-\frac{\sqrt{7}}{3}\cos s - \frac{1}{3}\sin s\right)\vec{e}_3 \right)$$

for $\vec{\gamma}(s) = \big(\gamma_1(s), \gamma_2(s)\big)$. Find the Frenet 2-vectors and Frenet formulas for this curve by using the componentwise metric.

Solution. First, let us show that $\vec{\gamma}$ is a unit speed curve. Here, for spatial quaternionic curves $\gamma_1, \gamma_2 : I \subset \mathbb{R} \to \mathbb{H}_P$, we get,

$$\vec{\gamma}'(s) = \big(\gamma_1'(s), \gamma_2'(s)\big)$$
$$= \left(\left(\frac{1}{2\sqrt{2}} - \frac{\sqrt{3}}{2\sqrt{2}}\cos s\right)\vec{e}_1 + \left(\frac{1}{2\sqrt{2}} + \frac{\sqrt{3}}{2\sqrt{2}}\cos s\right)\vec{e}_2 + \frac{\sqrt{3}}{2}\sin s\,\vec{e}_3, \right.$$
$$\left. \left(-\frac{1}{3}\sin s - \frac{\sqrt{7}}{3}\cos s\right)\vec{e}_1 + \frac{1}{3}\vec{e}_2 + \left(\frac{\sqrt{7}}{3}\sin s - \frac{1}{3}\cos s\right)\vec{e}_3 \right)$$

and by using (10),

$$\left[\|\vec{\gamma}'(s)\|_{\mathbb{H}_P^2}^{\mathbb{R}^2}\right]^2 = \langle \vec{\gamma}'(s), \vec{\gamma}'(s)\rangle_{\mathbb{H}_P^2}^{\mathbb{R}^2} = \vec{\gamma}'(s) \boxtimes \bar{\vec{\gamma}}'(s) = \big(\|\gamma_1'(s)\|_{\mathbb{H}_P}^2, \|\gamma_2'(s)\|_{\mathbb{H}_P}^2\big)$$
$$= \left(\frac{1}{8} - \frac{\sqrt{3}}{4}\cos s + \frac{3}{8}\cos^2 s + \frac{1}{8} + \frac{\sqrt{3}}{4}\cos s + \frac{3}{8}\cos^2 s + \frac{3}{4}\sin^2 s, \right.$$
$$\left. \frac{8}{9}\sin^2 s + \frac{2\sqrt{7}}{9}\sin s\cos s + \frac{8}{9}\cos^2 s + \frac{1}{9} - \frac{2\sqrt{7}}{9}\sin s\cos s \right)$$
$$= (1,1) = \vec{1}$$

is acquired. So, $\vec{\gamma}$ is a unit speed curve.

The unit tangent spatial quaternion 2-vector of $\vec{\gamma}$ is written as,

$$\vec{t}(s) = \vec{\gamma}'(s) = \big(t_1(s), t_2(s)\big)$$

with the unit tangent vector t_1 of γ_1 and the unit tangent vector t_2 of γ_2 in $\mathbb{H}_P$. From here,

$$\vec{t}'(s) = \vec{\gamma}''(s) = \big(t_1'(s), t_2'(s)\big)$$
$$= \left(\frac{\sqrt{3}}{2\sqrt{2}}\sin s\,\vec{e}_1 - \frac{\sqrt{3}}{2\sqrt{2}}\sin s\,\vec{e}_2 + \frac{\sqrt{3}}{2}\cos s\,\vec{e}_3, \left(-\frac{1}{3}\cos s + \frac{\sqrt{7}}{3}\sin s\right)\vec{e}_1 \right.$$
$$\left. + \left(\frac{\sqrt{7}}{3}\cos s + \frac{1}{3}\sin s\right)\vec{e}_3 \right)$$

and

$$\left\|\vec{t}'(s)\right\|_{\mathbb{H}_P^2}^{\mathbb{R}^2} = (\|t_1'(s)\|, \|t_2'(s)\|)$$
$$= \left(\sqrt{\frac{3}{8}\sin^2 s + \frac{3}{8}\sin^2 s + \frac{3}{4}\cos^2 s}, \right.$$

$$\left.\sqrt{\frac{1}{9}\cos^2 s-\frac{2\sqrt{7}}{9}\cos s\sin s+\frac{7}{9}\sin^2 s+\frac{7}{9}\cos^2 s+\frac{2\sqrt{7}}{9}\cos s\sin s+\frac{1}{9}\sin^2 s}\right)$$
$$=\left(\frac{\sqrt{3}}{2},\frac{2\sqrt{2}}{3}\right)$$

is calculated. By the definition $\vec{n}(s)=\vec{t}'(s)\boxdot_P\left[\left\|\vec{t}'(s)\right\|_{\mathbb{H}_P^n}^{\mathbb{R}^n}\right]^{-1}$,

$$\vec{n}(s)=\left(\frac{1}{\sqrt{2}}\sin s\,\vec{e}_1-\frac{1}{\sqrt{2}}\sin s\,\vec{e}_2+\cos s\,\vec{e}_3,\left(-\frac{1}{2\sqrt{2}}\cos s+\frac{\sqrt{7}}{2\sqrt{2}}\sin s\right)\vec{e}_1\right.$$
$$\left.+\left(\frac{\sqrt{7}}{2\sqrt{2}}\cos s+\frac{1}{2\sqrt{2}}\sin s\right)\vec{e}_3\right)$$

and

$$\vec{n}'(s)=\left(\frac{1}{\sqrt{2}}\cos s\,\vec{e}_1-\frac{1}{\sqrt{2}}\cos s\,\vec{e}_2-\sin s\,\vec{e}_3,\left(\frac{1}{2\sqrt{2}}\sin s+\frac{\sqrt{7}}{2\sqrt{2}}\cos s\right)\vec{e}_1\right.$$
$$\left.+\left(-\frac{\sqrt{7}}{2\sqrt{2}}\sin s+\frac{1}{2\sqrt{2}}\cos s\right)\vec{e}_3\right)$$

is obtained.

Finally, since $\vec{b}(s)=\vec{t}(s)\boxtimes\vec{n}(s)$,

$$\vec{b}(s)=\left(\left(\frac{\sqrt{3}}{2\sqrt{2}}+\frac{1}{2\sqrt{2}}\cos s\right)\vec{e}_1+\left(\frac{\sqrt{3}}{2\sqrt{2}}-\frac{1}{2\sqrt{2}}\cos s\right)\vec{e}_2\right.$$
$$+\left(-\frac{1}{2}\sin s\right)\vec{e}_3,\left(\frac{\sqrt{7}}{6\sqrt{2}}\cos s+\frac{1}{6\sqrt{2}}\sin s\right)\vec{e}_1+\frac{4}{3\sqrt{2}}\vec{e}_2$$
$$\left.+\left(\frac{1}{6\sqrt{2}}\cos s-\frac{\sqrt{7}}{6\sqrt{2}}\sin s\right)\vec{e}_3\right)$$

and

$$\vec{b}'(s)=\left(\left(-\frac{1}{2\sqrt{2}}\sin s\right)\vec{e}_1+\left(\frac{1}{2\sqrt{2}}\sin s\right)\vec{e}_2+\left(-\frac{1}{2}\cos s\right)\vec{e}_3,\right.$$
$$\left.\left(-\frac{\sqrt{7}}{6\sqrt{2}}\sin s+\frac{1}{6\sqrt{2}}\cos s\right)\vec{e}_1+\left(-\frac{1}{6\sqrt{2}}\sin s-\frac{\sqrt{7}}{6\sqrt{2}}\cos s\right)\vec{e}_3\right)$$

is evaluated. The curvature and torsion of this curve are obtained as,

$$\vec{\kappa}(s)=\left\|\vec{t}'(s)\right\|_{\mathbb{H}_P^2}^{\mathbb{R}^2}=\left(\frac{\sqrt{3}}{2},\frac{2\sqrt{2}}{3}\right),$$

$$\vec{\tau}(s)=\left\langle\vec{n}',\vec{b}\right\rangle_{\mathbb{H}_P^2}^{\mathbb{R}^2}=\left(\frac{1}{2},\frac{1}{3}\right).$$

As a result of these equations, the relationship between the Frenet 2-vectors of $\vec{\gamma}$ provides the equations given by Theorem 9, Corollary 10, and 11.

3.2. QUATERNIONIC CURVES IN $\mathbb{H}^n$

In this section, first, the definition of a quaternionic curve in $\mathbb{H}^n$ will be given. Then, regarding this definition, Frenet-Serret formulas will be obtained.

Definition 7. Let

$$\mathbb{H}^n = \left\{\vec{\beta} = (\beta_1, \beta_2, \dots, \beta_n) \ \mid \ \beta_i = \beta_{i0}\vec{e}_0 + \beta_{i1}\vec{e}_1 + \beta_{i2}\vec{e}_2 + \beta_{i3}\vec{e}_3 \in \mathbb{H}\right\}$$

be quaternion n-vectors space, where $1 \leq i \leq n$. In this space, the function,

$$\begin{aligned} \vec{\beta}: I \subset \mathbb{R} \to \mathbb{H}^n \\ s \to \vec{\beta}(s) = \big(\beta_1(s), \beta_2(s), \dots, \beta_n(s)\big) \\ = \left(\sum_{j=0}^{3} \beta_{1j}(s)\vec{e}_j, \sum_{j=0}^{3} \beta_{2j}(s)\vec{e}_j, \dots, \sum_{j=0}^{3} \beta_{nj}(s)\vec{e}_j\right) \\ = \left(\sum_{j=0}^{3} \beta_{ij}(s)\vec{e}_j\right)_{i=1}^{n} \end{aligned}$$

is called *n-dimensional quaternionic curve* for $j = 0,1,2,3$ where, $s \in I$ is an arc-length parameter. Here, each function β_i $(1 \leq i \leq n)$ are quaternionic curves in $\mathbb{H}$.

Definition 8. Let $\vec{\beta}: I \subset \mathbb{R} \to \mathbb{H}^n$ be the n-dimensional quaternionic curve with $I \subset \mathbb{R}$ unit interval and arc-length parameter $s \in I$ in $\mathbb{H}^n$. If,

$$\begin{aligned} \left[\left\|\vec{\beta}'(s)\right\|_{\mathbb{H}^n}^{\mathbb{R}^n}\right]^2 = \left\langle\vec{\beta}'(s), \vec{\beta}'(s)\right\rangle_{\mathbb{H}^n}^{\mathbb{R}^n} = \frac{1}{2}\left[\left(\vec{\beta}'(s) \boxtimes \overline{\vec{\beta}'}(s)\right) \boxplus \left(\vec{\beta}'(s) \boxtimes \overline{\vec{\beta}'}(s)\right)\right] \\ = \vec{\beta}'(s) \boxtimes \overline{\vec{\beta}'}(s) = \vec{1} \end{aligned}$$

then, $\vec{\beta}$ is called *n-dimensional unit speed quaternionic curve* where, $\vec{\beta}'(s)$ is the n-vector at each point of $\vec{\beta}$.

Definition 9. Let $\vec{\beta}: \mathrm{I} \subset \mathbb{R} \to \mathbb{H}^{\mathrm{n}}$ be n-dimensional unit speed quaternionic curve in $\mathbb{H}^n$ $\left(\text{i.e.} \left\|\vec{\beta}'(s)\right\|_{\mathbb{H}^n}^{\mathbb{R}^n} \neq \vec{0}\right)$. The n-vector $\vec{T}(s) = \vec{\beta}'(s) = \left(\sum_{j=0}^{3} \beta'_{ij}(s)\vec{e}_j\right)_{i=1}^{n}$ is called the unit tangent n-vector and $\vec{N}(s) = \vec{T}'(s) \boxdot \left[\left\|\vec{T}'(s)\right\|_{\mathbb{H}^n}^{\mathbb{R}^n}\right]^{-1}$ is called the unit normal n-vector of $\vec{\beta}$.

Theorem 12. Let $\vec{\beta}: I \subset \mathbb{R} \to \mathbb{H}^n$ be n-dimensional unit speed quaternionic curve in $\mathbb{H}^n$. $\vec{T}$ and $\vec{T}'$ are orthogonal, and also, $\vec{N} \boxtimes \overline{\vec{T}}$ is a quaternion n-vector.

Notation 1. Let $\vec{\beta}: I \subset \mathbb{R} \to \mathbb{H}^n$ be n-dimensional unit speed quaternionic curve given in $\mathbb{H}^n$. Since $\vec{T}$ and $\vec{N}$ are unit quaternion n-vectors, when $\vec{t} = \vec{N} \boxtimes \overline{\vec{T}}$ is selected,

$$\left[\left\|\vec{t}(s)\right\|_{\mathbb{H}^n}^{\mathbb{R}^n}\right]^2 = \vec{t}(s) \boxtimes \overline{\vec{t}}(s) = \left(\vec{N} \boxtimes \overline{\vec{T}}\right) \boxtimes \overline{\left(\vec{N} \boxtimes \overline{\vec{T}}\right)} = \vec{N} \boxtimes \left(\left(\overline{\vec{T}} \boxtimes \vec{T}\right) \boxtimes \overline{\vec{N}}\right) = \vec{N} \boxtimes \overline{\vec{N}}$$

$$= \vec{1}$$

is obtained. Hence $\vec{t}$ is also a unit quaternion n-vector and from the equation,

$$\vec{t} \boxtimes \vec{T} = \left(\vec{N} \boxtimes \overline{\vec{T}}\right) \boxtimes \vec{T} = \left(\vec{N} \boxtimes \vec{T}^{-1}\right) \boxtimes \vec{T} = \vec{N},$$

it is possible to take,

$$\vec{N} = \vec{t} \boxtimes \vec{T}. \tag{27}$$

Definition 10. Let a quaternionic curve $\vec{\beta}: I \subset \mathbb{R} \to \mathbb{H}^n$ be given, with unit interval $I \subset \mathbb{R}$ and arc-length parameter $s \in I$. Let us define the unit quaternion n-vector,

$$\vec{N}(s) = \vec{T}'(s) \boxdot \left[\left\|\vec{T}'(s)\right\|_{\mathbb{H}^n}^{\mathbb{R}^n}\right]^{-1} \tag{28}$$

with the unit tangent quaternion n-vector of $\vec{\beta}$ being $\vec{T}(s) = \vec{\beta}'(s) = \left(\sum_{j=0}^{3} \beta_{ij}'(s)\vec{e}_j\right)_{i=1}^{n}$. Here $\vec{T}$ and $\vec{T}'$ are orthogonal, so it is clear that, $\left\langle \vec{T}(s), \vec{N}(s)\right\rangle_{\mathbb{H}^n}^{\mathbb{R}^n} = \vec{0}$.

Lemma 13. Now, if a choice of $\vec{B}(s) = \vec{n}(s) \boxtimes \vec{T}(s)$ is made, the following are provided:
(i) $\vec{B}$ is the unit quaternion n-vector:

$$\left\langle \vec{B}, \vec{B}\right\rangle_{\mathbb{H}^n}^{\mathbb{R}^n} = \vec{B} \boxtimes \overline{\vec{B}} = \left(\vec{n} \boxtimes \vec{T}\right) \boxtimes \left(\overline{\vec{n} \boxtimes \vec{T}}\right) = \left(\vec{n} \boxtimes \vec{T}\right) \boxtimes \left(\overline{\vec{T}} \boxtimes \overline{\vec{n}}\right) = \vec{n} \boxtimes \left(\left(\vec{T} \boxtimes \overline{\vec{T}}\right) \boxtimes \overline{\vec{n}}\right)$$
$$= \vec{n} \boxtimes \overline{\vec{n}} = \vec{1}$$

(ii) $\left\langle \vec{T}, \vec{N}\right\rangle_{\mathbb{H}^n}^{\mathbb{R}^n} = \vec{0}$
(iii) $\left\langle \vec{T}, \vec{B}\right\rangle_{\mathbb{H}^n}^{\mathbb{R}^n} = \vec{0}$
(iv) $\left\langle \vec{N}, \vec{B}\right\rangle_{\mathbb{H}^n}^{\mathbb{R}^n} = \vec{0}$

Hence, the quaternion n-vector system $\{\vec{T}, \vec{N}, \vec{B}\}$ is orthogonal. Similarly, if a choice of $\vec{D}(s) = \vec{b}(s) \boxtimes \vec{T}(s)$ is made, the following are provided:
(i) $\vec{D}$ is the unit quaternion n-vector:

$$\left\langle \vec{D}, \vec{D}\right\rangle_{\mathbb{H}^n}^{\mathbb{R}^n} = \vec{D} \boxtimes \overline{\vec{D}} = \left(\vec{b} \boxtimes \vec{T}\right) \boxtimes \left(\overline{\vec{b} \boxtimes \vec{T}}\right) = \left(\vec{b} \boxtimes \vec{T}\right) \boxtimes \left(\overline{\vec{T}} \boxtimes \overline{\vec{b}}\right) = \vec{b} \boxtimes \left(\left(\vec{T} \boxtimes \overline{\vec{T}}\right) \boxtimes \overline{\vec{b}}\right)$$
$$= \vec{b} \boxtimes \overline{\vec{b}} = \vec{1}$$

(ii) $\left\langle \vec{T}, \vec{D}\right\rangle_{\mathbb{H}^n}^{\mathbb{R}^n} = \vec{0}$
(iii) $\left\langle \vec{N}, \vec{D}\right\rangle_{\mathbb{H}^n}^{\mathbb{R}^n} = \vec{0}$
(iv) $\left\langle \vec{B}, \vec{D}\right\rangle_{\mathbb{H}^n}^{\mathbb{R}^n} = \vec{0}$

Thus, the quaternion n-vector system $\{\vec{T}, \vec{N}, \vec{B}, \vec{D}\}$ is orthogonal. The $\vec{T}(s)$, $\vec{N}(s)$, $\vec{B}(s)$, and $\vec{D}(s)$ n-vectors are also called *Frenet n-vectors of the unit speed curve* in $\mathbb{H}^n$.

Definition 11. Curvature of $\vec{\beta}$ on $\mathbb{H}^n$ is defined as

$$\vec{K}(s) = \left\|\vec{T}'(s)\right\|_{\mathbb{H}^n}^{\mathbb{R}^n}. \tag{29}$$

Theorem 14. Let the unit tangent n-vector of the unit speed curve $\vec{\beta}$ be $\vec{T}$, and the curvature be $\vec{K}$ on $\mathbb{H}^n$. Then the following equality is achieved,

$$\vec{T}'(s) = \vec{N}(s) \boxdot \vec{K}(s). \tag{30}$$

Theorem 15. Let for every $s \in I$ of $\vec{\beta}$, the system $\{\vec{T}(s),\ \vec{N}(s),\ \vec{B}(s),\ \vec{D}(s)\}$ at the point $\vec{\beta}(s)$ and its curvatures $\vec{\kappa}(s)$, $\vec{K}(s)$, and $\left(\vec{\tau}(s) - \vec{K}(s)\right)$ are given. The derivatives of $\vec{T}'(s)$, $\vec{N}'(s)$, $\vec{B}'(s)$, and $\vec{D}'(s)$ n-vectors and the relationship between the curvatures are given by,

$$\begin{aligned}
&\vec{T}'(s) = \vec{N}(s) \boxdot \vec{K}(s)\\
&\vec{N}'(s) = \left(\vec{T}(s) \boxdot -\vec{K}(s)\right) \boxplus \left(\vec{B}(s) \boxdot \vec{\kappa}(s)\right)\\
&\vec{B}'(s) = \left(\vec{N}(s) \boxdot -\vec{\kappa}(s)\right) \boxplus \left(\vec{D}(s) \boxdot \left(\vec{\tau}(s) - \vec{K}(s)\right)\right)\\
&\vec{D}'(s) = \vec{B}(s) \boxdot -\left(\vec{\tau}(s) - \vec{K}(s)\right).
\end{aligned}$$

Proof: It has been shown that, $\vec{T}'(s) = \vec{N}(s) \boxdot \vec{K}(s)$ by Theorem 14.

By taking the derivative of (27) and considering (19) and taking (30) into account,

$$\vec{N}' = \left(\vec{t}' \boxtimes \vec{T}\right) \boxplus \left(\vec{t} \boxtimes \vec{T}'\right)$$

and

$$\vec{N}' = \left((\vec{n} \boxdot_P \vec{\kappa}) \boxtimes \vec{T}\right) \boxplus \left(\vec{t} \boxtimes \left(\vec{N} \boxdot \vec{K}\right)\right)$$

can be written. Using (27), $\vec{t} \boxtimes \vec{N} = -\vec{T}$ can also be written, in this case,

$$\vec{N}' = \left(\vec{T} \boxdot -\vec{K}\right) \boxplus \left(\vec{B} \boxdot \vec{\kappa}\right) \tag{31}$$

is obtained.

Now, taking the derivative of $\vec{B} = \vec{n} \boxtimes \vec{T}$ and by using (24) and (30),

$$\vec{B}' = \left(\vec{n}' \boxtimes \vec{T}\right) \boxplus \left(\vec{n} \boxtimes \vec{T}'\right)$$

and

$$\vec{B}' = \left(\left(\left(\vec{t} \boxdot_P - \vec{\kappa}\right) \boxplus \left(\vec{b} \boxdot_P \vec{\tau}\right)\right) \boxtimes \vec{T}\right) \boxplus \left(\vec{n} \boxtimes \left(\vec{N} \boxdot \vec{K}\right)\right)$$

obtained. From here,

$$\vec{B}' = \left(\left(\left(\vec{t} \boxtimes \vec{T}\right) \boxdot -\vec{\kappa}\right) \boxplus \left(\left(\vec{b} \boxtimes \vec{T}\right) \boxdot \vec{\tau}\right)\right) \boxplus \left(\left(\vec{n} \boxtimes \vec{N}\right) \boxdot \vec{K}\right)$$

is acquired. By using (27) and $\vec{n} \boxtimes \vec{N} = -\vec{D}$, if the equality is arranged,

$$\vec{B}' = \left(\vec{N} \boxdot -\vec{\kappa}\right) \boxplus \left(\vec{D} \boxdot \left(\vec{\tau} - \vec{K}\right)\right) \tag{32}$$

is obtained.

Finally, if the derivative of $\vec{D}=\vec{b}\boxtimes\vec{T}$ is taken and (26) and (30) are used,

$$\vec{D}'=\left(\vec{b}'\boxtimes\vec{T}\right)\boxplus\left(\vec{b}\boxtimes\vec{T}'\right)$$

and

$$\vec{D}'=\left((\vec{n}\boxdot_P-\vec{\tau})\boxtimes\vec{T}\right)\boxplus\left(\vec{b}\boxtimes\left(\vec{N}\boxdot\vec{K}\right)\right)$$

obtained. Thus,

$$\vec{D}'=\left(\left(\vec{n}\boxtimes\vec{T}\right)\boxdot-\vec{\tau}\right)\boxplus\left(\left(\vec{b}\boxtimes\vec{N}\right)\boxdot\vec{K}\right)$$

can be written. Considering the equations, $\vec{B}=\vec{n}\boxtimes\vec{T}$ and $\vec{b}\boxtimes\vec{N}=\vec{B}$, if the equality is rearranged,

$$\vec{D}'=\vec{B}\boxdot-\left(\vec{\tau}-\vec{K}\right) \tag{33}$$

is obtained.

Example 2. Let the curve $\vec{\beta}:I\subset\mathbb{R}\to\mathbb{H}^2$ be given as

$$\vec{\beta}(s)=\left(\frac{1}{2\sqrt{2}}\cos s\,\vec{e}_0+\frac{1}{2}\sin s\,\vec{e}_1+\frac{\sqrt{3}}{2}s\vec{e}_2+\frac{1}{2\sqrt{2}}\cos s\,\vec{e}_3,\frac{1}{3}\sin s\,\vec{e}_0-\frac{1}{3}s\vec{e}_1-\frac{1}{3}\cos s\,\vec{e}_2+\frac{\sqrt{7}}{3}s\vec{e}_3\right)$$

for $\vec{\beta}(s)=\left(\beta_1(s),\beta_2(s)\right)$. Find the Frenet 2-vectors and Frenet formulas for this curve.

Solution. First, let us show that $\vec{\beta}$ has a unit speed. Here, for the quaternionic curves $\beta_1,\beta_2:I\subset\mathbb{R}\to\mathbb{H}$,

$$\vec{\beta}'(s)=\left(\beta_1'(s),\beta_2'(s)\right)$$
$$=\left(-\frac{1}{2\sqrt{2}}\sin s\,\vec{e}_0+\frac{1}{2}\cos s\,\vec{e}_1+\frac{\sqrt{3}}{2}\vec{e}_2-\frac{1}{2\sqrt{2}}\sin s\,\vec{e}_3,\frac{1}{3}\cos s\,\vec{e}_0-\frac{1}{3}\vec{e}_1+\frac{1}{3}\sin s\,\vec{e}_2+\frac{\sqrt{7}}{3}\vec{e}_3\right)$$

is evaluated, and

$$\left[\left\|\vec{\beta}'(s)\right\|_{\mathbb{H}^2}^{\mathbb{R}^2}\right]^2=\left\langle\vec{\beta}'(s),\vec{\beta}'(s)\right\rangle_{\mathbb{H}^2}^{\mathbb{R}^2}=\vec{\beta}'(s)\boxtimes\bar{\vec{\beta}}'(s)=(\|\beta_1'(s)\|_{\mathbb{H}}^2,\|\beta_2'(s)\|_{\mathbb{H}}^2)$$
$$=\left(\frac{1}{8}\sin^2 s+\frac{1}{4}\cos^2 s+\frac{3}{4}+\frac{1}{8}\sin^2 s,\frac{1}{9}\cos^2 s+\frac{1}{9}+\frac{1}{9}\sin^2 s+\frac{7}{9}\right)=(1,1)=\vec{1}$$

is acquired. So, $\vec{\beta}$ is a unit speed curve.

With the unit tangent vector of β_1 being T_1, and the unit tangent vector of β_2 being T_2, the unit tangent quaternion 2-vector of $\vec{\beta}$ is,

$$\vec{T}(s)=\vec{\beta}'(s)=\left(T_1(s),T_2(s)\right).$$

With this,

$$\vec{T}'(s)=\vec{\beta}''(s)=\left(T_1'(s),T_2'(s)\right)$$
$$=\left(-\frac{1}{2\sqrt{2}}\cos s\,\vec{e}_0-\frac{1}{2}\sin s\,\vec{e}_1-\frac{1}{2\sqrt{2}}\cos s\,\vec{e}_3,-\frac{1}{3}\sin s\,\vec{e}_0+\frac{1}{3}\cos s\,\vec{e}_2\right)$$

is found. Then, the curvature of $\vec{\beta}$,

$$\vec{K}(s) = \left\|\vec{T}'(s)\right\|_{\mathbb{H}^2}^{\mathbb{R}^2} = (\|T_1'(s)\|_{\mathbb{H}}, \|T_2'(s)\|_{\mathbb{H}})$$
$$= \left(\sqrt{\frac{1}{8}\cos^2 s + \frac{1}{4}\sin^2 s + \frac{1}{8}\cos^2 s}, \sqrt{\frac{1}{9}\sin^2 s + \frac{1}{9}\cos^2 s}\right) = \left(\frac{1}{2}, \frac{1}{3}\right)$$

is calculated in $\mathbb{H}^n$. From the definition $\vec{N}(s) = \vec{T}'(s) \boxdot \left[\left\|\vec{T}'(s)\right\|_{\mathbb{H}^n}^{\mathbb{R}^n}\right]^{-1}$,

$$\vec{N}(s) = \left(-\frac{1}{2\sqrt{2}}\cos s\,\vec{e}_0 - \frac{1}{2}\sin s\,\vec{e}_1 - \frac{1}{2\sqrt{2}}\cos s\,\vec{e}_3, -\frac{1}{3}\sin s\,\vec{e}_0 + \frac{1}{3}\cos s\,\vec{e}_2\right) \boxdot_P \left(\frac{1}{2}, \frac{1}{3}\right)^{-1}$$
$$= \left(-\frac{1}{\sqrt{2}}\cos s\, e_0 - \sin s\, e_1 - \frac{1}{\sqrt{2}}\cos s\, e_3, -\sin s\, e_0 + \cos s\, e_2\right)$$

is obtained.

Also, since $\vec{n}$ and $\vec{b}$ are Frenet n-vectors of $\vec{\gamma}$ in Example 1, this provides $\vec{t} = \vec{N} \boxtimes \overline{\vec{T}}$ as a unit tangent quaternion n-vector and for $\vec{B}(s) = \vec{n}(s) \boxtimes \vec{T}(s)$ and $\vec{D}(s) = \vec{b}(s) \boxtimes \vec{T}(s)$ we get:

$$\vec{B}(s) = \vec{n}(s) \boxtimes \vec{T}(s)$$
$$= \left(\frac{\sqrt{3}}{2\sqrt{2}}\sin s\, e_0 - \frac{\sqrt{3}}{2}\cos s\, e_1 + \frac{1}{2}e_2 + \frac{\sqrt{3}}{2\sqrt{2}}\sin s\, e_3, -\frac{4}{3\sqrt{2}}\cos s\, e_0 - \frac{1}{6\sqrt{2}}e_1 - \frac{4}{3\sqrt{2}}\sin s\, e_2 + \frac{\sqrt{7}}{6\sqrt{2}}e_3\right)$$

and

$$\vec{D}(s) = \vec{b}(s) \boxtimes \vec{T}(s) = \left(-\frac{1}{\sqrt{2}}e_0 + \frac{1}{\sqrt{2}}e_3, \frac{\sqrt{7}}{2\sqrt{2}}e_1 + \frac{1}{2\sqrt{2}}e_3\right).$$

As a result, an orthogonal quaternion n-vector system $\{\vec{T}, \vec{N}, \vec{B}, \vec{D}\}$ is obtained. Also, if the curvatures $\vec{\kappa}(s) = \left(\frac{\sqrt{3}}{2}, \frac{2\sqrt{2}}{3}\right)$ and $\vec{\tau}(s) = \left(\frac{1}{2}, \frac{1}{3}\right)$ of $\vec{\gamma}$ in Example 1 are used, the Frenet formulas of $\vec{\beta}$ in Theorem 15 are provided.

4. CONCLUSIONS

Frenet-Serret formulas in quaternionic space have been studied in different sources. One of them is the study [14], which uses a unique methodology. In light of this study, we investigated the Frenet-Serret formulas for the n-dimensional quaternionic space. In order to achieve this, we first needed to define a metric on $\mathbb{R}^n$, which we processed in our previous study [11]. And thus, we have established the module structure of $\mathbb{H}^n$ over $\mathbb{R}^n$. Also, we have given real variable quaternion-valued vector functions for curve definition in n-dimensional quaternion space.

Furthermore, with this work, we have structured the $\{\vec{t}, \vec{n}, \vec{b}\}$ and $\{\vec{T}, \vec{N}, \vec{B}, \vec{D}\}$ frameworks in both $\mathbb{H}_P^n$ and $\mathbb{H}^n$ spaces, respectively, and obtained the curvature and the torsion formulations. We supported our work by giving examples of the spaces $\mathbb{H}_p^n$ and $\mathbb{H}^n$. As a result, we have successfully constituted the basis that will facilitate n-dimensional quaternionic space and help us to obtain different formulas.

Geometrically, the derivation of the new curvature and torsion vectors, $\vec{\kappa}$ and $\vec{\tau}$, as n-dimensional real vectors rather than real scalars is a direct and powerful consequence of employing the componentwise metric. Physically, this implies that in a complex configuration space representing a multi-agent system, the kinematic behavior, specifically the bending and twisting of the trajectory, of each individual agent is preserved and can be monitored independently. A standard global metric would compress this rich geometric data into a single scalar, causing information loss regarding the individual components. Thus, the vector-valued Frenet elements introduced in this study provide a decentralized yet unified mathematical tool for analyzing simultaneous multi-trajectory tracking and multi-body kinematics. In conclusion, the framework established in this study sheds light on future research regarding the kinematics of curves in advanced generalized spaces, such as quaternionic curvature theory, the geometric properties of quaternionic surfaces, and Minkowski space.

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