ORIGINAL PAPER

EXPANSION OF INTERPOLATION IN THE OCTONIONS CONTEXT

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Abstract. *This paper extends the notion of interpolation from the quaternionic framework to the octonionic setting. By establishing a well-adapted algebraic basis, we examine how replacing the quaternion space with octonions influences the generated curves. The obtained curves in the octonionic space are not only differentiable but also display smoother behavior compared to those generated by corresponding methods in the quaternionic case. Moreover, the convergence is achieved with less iteration.*
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1. INTRODUCTION

Interpolation is a method for generating new data points from a given dataset. Essentially, interpolating schemes produce curves that model the trajectory of rigid motions, particularly rotational behaviour. Among the many mathematical tools used to represent rotations, Clifford algebras are regarded as one of the most efficient structures [1]. It is worth noting that Clifford algebras are essential in computer graphics, robotics, geometric modelling, neural networks, and several other application areas.

In 1985, Shoemake proposed employing quaternions in visual computing and introduced the slerp curve, which corresponds to geodesic segments from a differential-geometric viewpoint [2]. He later constructed cubic Bézier curves on the quaternionic sphere via the de Casteljau scheme with six line segments [3]. Schlge continued this direction by applying Barry and Goldman's recursive algorithm using six slerp operations to construct B-splines, cardinal interpolation splines, and Bézier curves on the quaternion sphere; however, this approach did not incorporate all input control points [4,5]. Nielson and Heiland introduced an alternative to this limitation, but their method was time-consuming and ineffective for motion graphics use [6].

To enhance performance, Shoemake [2] introduced the squad curve, which is created by composing three slerp operations. Although more computationally efficient than the Bézier technique, the generated curve lacked sufficient smoothness. Follow-up work by Kim et al. concentrated on smoothing the squad curve and, in a separate study, explored the development of circular quaternion curves. Nielson et al. later put forward an additional method distinct from the earlier methods [7-9]. Through continued progress, this evolutionary process led to the construction of various quaternion-based curves under the framework of Clifford algebras [10]. This paper addresses the following main question: In what way does

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changing the quaternion-based context to an octonion-based structure influence the interpolation process?

Section 2 provides the appropriate context for performing interpolation. Section 3 is dedicated to the use of octonions in interpolation. The last section compares the graphics obtained from quaternions and octonion spaces.

2. PRELIMINARIES

Octonion's 8-dimensional vector space is a normed division algebra that lacks properties of multiplicative commutation and associativity. The members of the octonion space O can be considered as follows:

$$x = x_0e_0 + x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4 + x_5e_5 + x_6e_6 + x_7e_7 \quad (1)$$

where $\{e_0, e_1, ..., e_7\}$ are the orthonormal basis?

The conjugate of x is given by

$$x^* = x_0e_0 - x_1e_1 - x_2e_2 - x_3e_3 - x_4e_4 + x_5e_5 - x_6e_6 - x_7e_7, \quad (2)$$

$$(xy)^* = y^*x^*. \quad (3)$$

From these descriptions, every octonion has a multiplicative inverse and norm as follows:

$$y^{-1} = \frac{y^*}{\|y\|}, \quad (4)$$

$$\|y\| = \sqrt{y^*y}, \quad \|xy\| = \|x\|\,\|y\|. \quad (5)$$

Also

$$e_ie_j = -e_je_i, \quad (6)$$

$$(e_ie_j)e_k = -e_i(e_je_k). \quad (7)$$

Definition 1. Let an octonion $x = [a,b] = [a,(a_1,...,a_7)]$ is a pair consisting of a scalar component $a \in \mathbb{R}$ and a vector component $(a_1,\ldots,a_7) \in \mathbb{R}^7$.

Using the 7-dimensional cross product $x \times y = \frac{1}{2}(xy - yx)$. The multiplication of two octonion members is considered as follows:

$$xy = [a,b][a',b'] = [aa' - bb', ab' + a'b + 2b \times b'] \quad (8)$$

where $y = [a',b']$ is an octonion.

Definition 2. Let $x \in O$, if $\|x\| = 1$, x is called a unit octonion.

Each unit member of the octonion space can be considered as follows:

$$x = [a,b] = [a,(a_1,...,a_7)] \Rightarrow \exists b \in \mathbb{R}^7, \theta \in (-\pi,\pi) \text{ such that } x = [cos\theta, bsin\theta]. \tag{9}$$

It is enough to check it in two conditions. The first case: $x = [1,0]$ then we suppose that $\theta = 0$ and b is a unit vector in $\mathbb{R}^7$. The second case: If $x \neq [1,0]$ we let $k = |a|$ and $b = \frac{1}{k}a \Rightarrow a = kb$, where b is a unit vector in $\mathbb{R}^7$ since

$$1 = \|x\|^2 = a^2 + k^2 bb = a^2 + K^2. \tag{10}$$

Also, for the unit circle $\cos^2\theta + \sin^2\theta = 1$.

For the unit octonion in the form $x = [\cos\theta, b\sin\theta]$. The logarithm function is considered as $\log x = [0, \theta b]$.

For every octonion in the form $x = [0, \theta b]$, $\theta \in \mathbb{R}$, $b \in \mathbb{R}^7$, $|b| = 1$ the exponential function is considered as

$$\exp x = [\cos\theta, b\sin\theta]. \tag{11}$$

For the unit octonion x and every $t \in \mathbb{R}$, we have

$$\begin{aligned} x^t &= \exp(t\log x), \\ log(x^t) &= t\log x, \\ x^c x^d &= x^{c+d}, \\ (x^c)^d &= x^{cd}. \end{aligned} \tag{12}$$

Lemma 1. Let be $h, p \in C^1(\mathbb{R}, O)$. Then

$$\begin{aligned} &\frac{d}{dt}\big(h(t)p(t)\big) = \frac{dh(t)}{dt}p(t) + h(t)\frac{dp(t)}{dt}, \\ &\frac{dh(p(t))}{dt} = h'(p(u))p'(u), \\ &\frac{d}{dt}x(t)^{v(t)} = [-\sin\big(v(t)\theta(t)\big)\big(v'(t)\theta(t) + v(t)\theta'(t)\big), \\ &\qquad \cos\big(v(t)\theta(t)\big)\big(v'(t)\theta(t) + v(t)\theta'(t)\big)b(t) + \sin\big(v(t)\theta(t)\big)b'(t)]. \end{aligned} \tag{13}$$

Proof:

$$\frac{d}{dt}\big(h(t)\,p(t)\big)=\lim_{\varepsilon\to 0}\frac{h(t+\varepsilon)\,p(t+\varepsilon)-h(t)\,p(t)}{\varepsilon}$$
$$=\lim_{\varepsilon\to 0}\frac{h(t+\varepsilon)\,p(t+\varepsilon)-h(t+\varepsilon)\,p(t)+h(t+\varepsilon)\,p(t)-h(t)\,p(t)}{\varepsilon}$$
$$=\lim_{\varepsilon\to 0}\left[h(t+\varepsilon)\frac{p(t+\varepsilon)-p(t)}{\varepsilon}+p(t)\frac{h(t+\varepsilon)-p(t)}{\varepsilon}\right]$$
$$=h(t)\,p'(t)+h(t)\,p'(t)$$

Also, for an arbitrary real number u

$$\frac{d}{du}h\big(p(u)\big)=\lim_{x\to u}\frac{h\big(p(x)\big)-h\big(p(u)\big)}{x-u}$$
$$=\lim_{x\to u}\left(\frac{h\big(p(x)\big)-h\big(p(u)\big)}{p(x)-p(u)}\cdot\frac{p(x)-p(u)}{x-u}\right)$$
$$=\lim_{p(x)\to p(u)}\frac{h\big(p(x)\big)-h\big(p(u)\big)}{p(x)-p(u)}\lim_{x\to u}\left(\frac{p(x)-p(u)}{x-u}\right)$$
$$=h'\big(p(u)\big)\,p'(u)$$

$$\frac{d}{dt}\Big(x(t)^{v(t)}\Big)=\frac{d}{dt}\exp\Big(v(t)\log\big(x(t)\big)\Big)$$
$$=\frac{d}{dt}\exp\Big(v(t)\big[0,b(t)\theta(t)\big]\Big)$$
$$=\frac{d}{dt}\exp\big[0,v(t)b(t)\theta(t)\big]$$
$$=\frac{d}{dt}\Big[\cos\big(v(t)\theta(t)\big),\sin\big(v(t)\theta(t)\big)b(t)\Big]$$
$$=[-\sin\big(v(t)\theta(t)\big)\big(v'(t)\theta(t)+v(t)\theta'(t)\big),$$
$$\cos\big(v(t)\theta(t)\big)\big(v'(t)\theta(t)+v(t)\theta'(t)\big)b(t)+\sin\big(v(t)\theta(t)\big)b'(t)].$$

In the 8-dimensional space of the octonions, using the octonion $x=[a,b]$ where b is a vector$\left(b\in\mathbb{R}^7\right)$ and a is a scalar $\left(a\in\mathbb{R}\right)$. A point p can be rotated around the axis b in $\mathbb{R}^8$. In fact, the rotation takes place in the unit sphere $S(1)$, which is obtained from intersection $\mathbb{R}^7\bigcap S(1)^7$. If $\{e_0,\ldots,e_7\}$ is a base for $\mathbb{R}^8$ then in this space, $span\{e_0,e_4\}$, $span\{e_2,e_1\}$, $span\{e_5,e_7\}$ and $span\{e_3,e_6\}$ are mutually perpendicular.

$$\varphi_x(p)=\frac{x^*px}{Q(x)}\leftrightarrow \textit{rotation of } b$$

where $Q(x)=xx^*$.

Theorem 1. Suppose that $x = cos\theta + bsin\theta$ be a unit octonion, and p is a point in octonion space. Then the 7-dim vector p rotates around the 7-dim axis b by an angle of 2θ, that is $\varphi_x(p) = x^* px$.

Proof: Using the proof requires showing that $\varphi_x(p)$ is a 7-dim vector, a length preserving function of 7-dim vectors, a linear transformation, and does not have a reflection component. Since p is a 7-dim vector in an 8-dim space. So, its vector component is zero. $\varphi_x(p)$ is a 7-dim vector, $S(p) = 0$ denoted by the scalar part of p

$$\begin{aligned} S(\varphi_x(p)) &= S(xpx^*) \\ &= \left[(xpx^*) + (xpx^*)^*\right]/2 \\ &= \left[xpx^* + xp^*x^*\right]/2 \\ &= x\left[(p + p^*)/2\right]x^* \\ &= xS(p)x^* \\ &= S(p) \\ &= 0. \end{aligned}$$

$\varphi_x(p)$ is length preserving, $\mathcal{N}\imath(p)$ denoted the norm of p

$$\begin{aligned} \mathcal{N}\imath(\varphi_x(p)) &= \mathcal{N}\imath(xpx^*) \\ &= \mathcal{N}\imath(x)\mathcal{N}\imath(p)\mathcal{N}\imath(x^*) \\ &= \mathcal{N}\imath(x)\mathcal{N}\imath(p)\mathcal{N}\imath(x) \\ &= \mathcal{N}\imath. \end{aligned}$$

$\varphi_x(p)$ is a linear transformation,

$$\begin{aligned} \varphi_x(wp + h) &= x(wp + h)x^* \\ &= (xwpx^*) + (xhx^*) \\ &= w(xpx^*) + (xhx^*) \\ &= w\varphi_x(p) + \varphi_x(h). \end{aligned}$$

Consider φ as a function of x for a fixed vector p. This function is continuous. For each x it is a linear transformation with a determinant $D(x)$, so the determinant itself is continuous. Thus, $\lim_{x\to1}\varphi(x) = \varphi(x) = I$, the identity function (the limit is taken along any path of octonions which approach the octonion 1) and $\lim_{x\to1} D(x) = D(1) = 1$. By continuity, $D(x)$ is identically 1 and $\varphi(x)$ does not have a reflection component.

The unit rotation axis is the 7-dim vector b and the rotation angle is 2θ. To see that b is a unit rotation axis; we need only show that b is unchanged by the rotation.

$$\begin{aligned}\varphi_x(b) &= xbx^* \\ &= (\cos\theta + b\sin\theta)b(\cos\theta - b\sin\theta) \\ &= (\cos\theta)^2 b - (\sin\theta)^2 b^3 \\ &= (\cos\theta)^2 b - (\sin\theta)^2(-b) \\ &= b,\end{aligned}$$

to prove the rotation angle is equal to 2θ.

According to what was said earlier $span\{e_0, e_4\}$ and $span\{e_5, e_7\}$ are perpendicular, assume that $b = e_4$ since $e_5 e_7 = e_4$, $e_7 e_4 e_7 = -e_5 e_7 = -e_4$. Then, by using the relation $x \cdot x' = \|x\| \, \|x'\| cos\phi$ so

$$\begin{aligned}\cos(\phi) = e_7 \cdot (xe_7 x) = S(e_7^* xe_7 qx^*) &= S\left[-e_7(\cos\theta + e_4\sin\theta)e_7(\cos\theta - e_4\sin\theta)\right] \\ &= S\left[(-e_7\cos\theta - e_7 e_4\sin\theta)(e_7\cos\theta - e_7 e_4\sin\theta)\right] \\ &= S\left[-e_7^2\cos^2\theta + e_7^2 e_4\sin\theta\cos\theta - e_7 e_4 e_7\sin\theta\cos\theta + (e_7 e_4)^2\sin^2\theta\right] \\ &= S\left[\cos^2\theta - \sin^2\theta - (e_4 + e_7 e_4 e_7)\sin\theta\cos\theta\right] \\ &= S\left[\cos^2\theta - \sin^2\theta - e_4(2\sin\theta\cos\theta)\right] \\ &= \cos^2\theta - \sin^2\theta \\ &= \cos(2\theta).\end{aligned}$$

Proposition 1. Octonions perform the same rotation on the line passing through the origin.

Proposition 2. Suppose that $x \in O_1$, $y = [c, db]$ where $c, d \in \mathbb{R}$ and $b \in \mathbb{R}^7$. If $x^*[c,b]x = [c,b']$, then $x^*[c,db]x = [c,db']$.

Proposition 3. Suppose that $x, y \in O_1$, $y = [cos\theta, sin\theta b]$, $t \in \mathbb{R}$. Then

$$x^* y^t x = (x^* yx)^t, \frac{d}{dt}x^t = x^t\log(x). \tag{14}$$

Proposition 4. Suppose that $x_1, x_2 \in O_1$. Consecutive rotations of a vector by x_1, x_2 are the same as rotations by $x_1 x_2$.

3. THE USE OF THE OCTONIONS IN INTERPOLATION

3.1. INTERPOLATION CURVE WITH ONLY TWO CONTROL POINTS

Regarding the interpolation on a straight line, the assumed vector e_i is the initial vector and e_j is the final vector with the parameter $t \in [0,1]$ determined by [7, 11, 12]:

$$\begin{aligned} z_t &= e_i + t\left(e_i - e_j\right) \\ &= \left(1-t\right)e_i + te_j . \end{aligned} \tag{15}$$

It is feasible to interpolate the movement along the curved path(an arc created by the intersection of a plane passing through the origin $\left(e_i \wedge e_j\right)$ with a sphere). To generalize this method to the octonionic sphere, one can use the octonion alternative property. In fact, one can consider the rotated version of the two mentioned vectors

$$z_t = \alpha_t e_i + \beta_t e_j \tag{16}$$

that is, e_j is translated to e_i by a rotor $\varphi_x\left(e_i\right)$. Because moving along the great circle is like moving from one point to another

$$e_j = e_i \varphi_x\left(e_i\right) = \left(\varphi_x\left(e_i\right)\right)^* e_i , \tag{17}$$

where $\left(\varphi_x\left(e_i\right)\right)^*$ is conjugate $\varphi_x\left(e_i\right)$.

So,

$$z_t = e_i\left(\varphi_x\left(e_i\right)\right)^t = e_i\left(y\right)^t = e_i\left(e^{\theta b'}\right)^t \tag{18}$$

where θ rotation angle from e_i to e_j, and $y \in O_1$. Using

$$e_i \wedge e_j = \frac{1}{2}\left(e_i e_j - e_j e_i\right) \tag{19}$$

That is why,

$$z_t = \alpha_t e_i + \beta_t e_j$$

$$e_i\left(\varphi_x\left(e_i\right)\right)^t = \alpha_t e_i + \beta_t e_i \varphi_x\left(e_i\right)$$

$$e_i\left(\varphi_x\left(e_i\right)\right)^t \wedge \partial e_i \varphi_x\left(e_i\right) = \left(\alpha_t e_i + \beta_t e_i \varphi_x\left(e_i\right)\right) \wedge e_i \varphi_x\left(e_i\right),$$

Therefore

$$\frac{1}{2}\left(e_i\left(\varphi_x\left(e_i\right)\right)^t e_i \varphi_x\left(e_i\right) - e_i \varphi_x\left(e_i\right) e_i\left(\varphi_x\left(e_i\right)^t\right)\right) = \frac{1}{2}\alpha_t\left(e_i^2\left(\varphi_x\left(e_i\right)\right) - e_i\left(\varphi_x\left(e_i\right)e_i\right)\right)$$

Hence we have

$$e_i^2\left(\varphi_x\left(e_i\right)\right)^t\varphi_x\left(e_i\right)^* - e_i^2\varphi_x\left(e_i\right)^*\varphi_x\left(e_i\right)^t = \alpha_t\left(e_i^2\varphi_x\left(e_i\right) - \partial\alpha_i^2\varphi_x\left(\partial e_i\right)^*\right)$$

Since $e_i^2 = 1$

$$\varphi_x\left(e_i\right)^{t*}\varphi_x\left(e_i\right) - \left(\varphi_x\left(e_i\right)\right)^*\varphi_x\left(e_i\right)^t = \alpha_t\left(\varphi_x\left(e_i\right) - \left(\varphi_x\left(e_i\right)\right)^*\right)$$

$$\varphi_x\left(e_i\right)^{t*}\varphi_x\left(e_i\right) - \left(\varphi_x\left(e_i\right)^{t*}\varphi_x\left(e_i\right)^t\right)^* = \alpha_t\left(\varphi_x\left(e_i\right) - \left(\varphi_x\left(e_i\right)\right)^*\right).$$

So

$$\alpha_t = \frac{\sin\left(\left(1-t\right)\theta\right)}{\sin\theta}.$$

In the same way

$$\beta_t = \frac{\sin\left(t\theta\right)}{\sin\theta}.$$

Therefore

$$z_t = \frac{\sin\left(\left(1-t\right)\theta\right)}{\sin\theta}e_i + \frac{\sin\left(t\theta\right)}{\sin\theta}e_j.$$

If the angle θ is very small

$$\left(\theta \approx 0 \Rightarrow sin\theta \approx \theta\right)$$

Then we have

$$z_t = \left(1-t\right)e_i + te_j.$$

So, in the octonion unit sphere, with only two control points, the beginning and the end, the interpolator curve is as follows:

$$Oslerp\left(y,x,t\right) = \frac{\sin\left(\left(1-t\right)\theta\right)}{\sin\theta}y + \frac{\sin\left(t\theta\right)}{\sin\theta}x. \tag{20}$$

In differential geometry, the great circle on a sphere is called a geodesic. This curve is equivalent to a straight line on the plane.

Lemma 2. If $x_1, x_2, y \in O$ then $\left(yx_1\right)\cdot\left(yx_2\right) = \|y\|^2\left(x_1 \cdot x_2\right)$.

Proof: Let

$$y = \left[s,v\right], x_1 = \left[a_1,\left(b_1,b_2,b_3\right)\right] = \left[a_1,b\right]$$

$$x_2 = \left[a_2,\left(b_1',b_2',b_3'\right)\right] = \left[a_2,b'v_2\right] \in O.$$

Using 7-dim cross product and

$$\left(v\times b\right)\cdot\left(v\times b'\right) = \left(v\cdot v\right)\left(b\cdot b'\right) - \left(v\cdot b'\right)\left(b\cdot v\right)$$

So

$$
\begin{aligned}
(yx_1)\cdot(yx_2) &= [sa_1 - v\cdot b, sb + a_1v + 2v\times b]\cdot[sa_2 - v\cdot b', sb' + a_2v + 2v\times b'] \\
&= s^2a_1a_2 - sa_1v\cdot b' - sa_2v\cdot b + (v\cdot b)(v\cdot b') \\
&\quad + s^2b\cdot b' + sa_2v\cdot b + 2sb\cdot(v\times b') \\
&\quad + sa_1b\cdot b' + a_1a_2v\cdot v + 2a_1v\cdot(v\times b') \\
&\quad + 2s(v\times v_1)\cdot b' + 2a_2(v\times b)\cdot v + 4(v\times b)\cdot(v\times b') \\
&= s^2a_1a_2 + (v\cdot v_1)(v\cdot v_2) \\
&\quad + s^2b\cdot v_2 + 2sb\cdot(v\times b') + a_1a_2v\cdot v \\
&\quad + 2s(v\times v_1)\cdot v_2 + 4(v\times v_1)\cdot(v\times v_2) \\
&= (s^2 + v\cdot v)a_1a_2 + (s^2 + v\cdot v)b\cdot b' \\
&\quad + 2sb\cdot(v\times b') + 2s(v\times b)\cdot b' \\
&= \|y\|^2(x_1\cdot x_2) + 2sb\cdot(v\times b') + 2sb'\cdot(v\times b) \\
&= \|y\|^2(x_1\cdot x_2).
\end{aligned}
$$

Proposition 5. The curve $Oslerp(y,x,t): O_1\times O_1\times[0,1]\to O_1$ on the octonion sphere is a geodesic curve with starting point y and the ending point x if and only if its second derivative vector is parallel to its position vector.

Proof: The following conditions must be met

$$
\begin{aligned}
&Oslerp(y,x,0) = y \\
&Oslerp(y,x,1) = x \\
&\|Oslerp(y,x,t)\| = 1,\ t\in[0,1] \\
&\frac{d^2}{dt^2}Oslerp(y,x,t) = c\,Oslerp(y,x,t),\ c\le 0\in\mathbb{R}\ .
\end{aligned}
$$

The first three relations are trivial

$$
\begin{aligned}
\frac{d}{dt}Oslerp(y,x,t) &= \frac{d}{dt}y\left(y^*x\right)^t \\
&= y\left(y^*x\right)^t\log\left(y^*x\right) \\
&= Oslerp(y,x,t)\log\left(y^*x\right). \\
\frac{d^2}{dt^2}Oslerp(y,x,t) &= y\left(y^*x\right)^t\log(y^*x)^2 \\
&= Oslerp(y,x,t)\log\left(y^*x\right)^2
\end{aligned}
$$

Since $y^*, x\in O_1$, then $\|y^*\|=1, \|x\|=1 \Rightarrow \|y^*x\| = \|y^*\|.\|x\| = 1$. $\exists\theta\in\mathbb{R}$ and $b\in\mathbb{R}^7$, $|b|=1$ s.t $y^*x = [\cos\theta, \sin\theta b]$.

Then

$$\log\left(y^{*}x\right)^{2}=\left[0,\theta b\right]^{2}=\left[-\theta^{2}b\cdot b,2\theta^{2}b\times b\right]=\left[-\theta^{2},0\right].$$

Thus, $\frac{d^{2}}{dt^{2}}Oslerp\left(y,x,t\right)=cOslerp\left(y,x,t\right)$ where $c=-\theta^{2}\leq 0$.

Proposition 6. The geodesic curve $Oslerp\left(y,x,t\right),t\in\left[0,1\right]$ is the shortest distance between the starting point y and the ending point x for any $y,x\in O_{1}$.

Proof: Similar to the process of proving Theorem 1, it suffices to find the angle between the initial vector y and the $Oslerp\left(y,x,\frac{1}{2}\right)$. With exp and log functions

$$\begin{aligned}
cos\left(\alpha\right)&=y\cdot\left(y\left(y^{*x}\right)\right)^{\frac{1}{2}}\\
&=y\cdot\left(y\left[\cos\left(\psi\right),\sin\left(\psi\right)w\right]\right)^{\frac{1}{2}}\\
&=y\cdot\left(y\exp\left(\frac{1}{2}\log[\cos\left(\psi\right),\sin\left(\psi\right)w]\right)\right)\\
&=y\cdot\left(y\exp\left([0,\left(\frac{\psi}{2}\right)w]\right)\right)\\
&=y\cdot\left(y[\cos\left(\frac{\psi}{2}\right),\sin\left(\frac{\psi}{2}\right)w]\right)\\
&=\left(y[1,0]\right)\cdot\left(p\left[\cos\left(\frac{\psi}{2}\right),\sin\left(\frac{\psi}{2}\right)w\right]\right)\\
&=\|y\|^{2}\left([1,0]\cdot\left[\cos\left(\frac{\psi}{2}\right),\sin\left(\frac{\psi}{2}\right)w\right]\right)=\|y\|^{2}\cos\left(\frac{\psi}{2}\right)=\cos\left(\frac{\psi}{2}\right).
\end{aligned}$$

Since $\psi\in\left(-\pi,\pi\right)$ yields $cos\left(\frac{\psi}{2}\right)\geq 0$ and so $\cos\left(\alpha\right)\geq 0$. The difference between the resulting graphs in octonion and quaternion spaces is noticeable in Fig. 1.

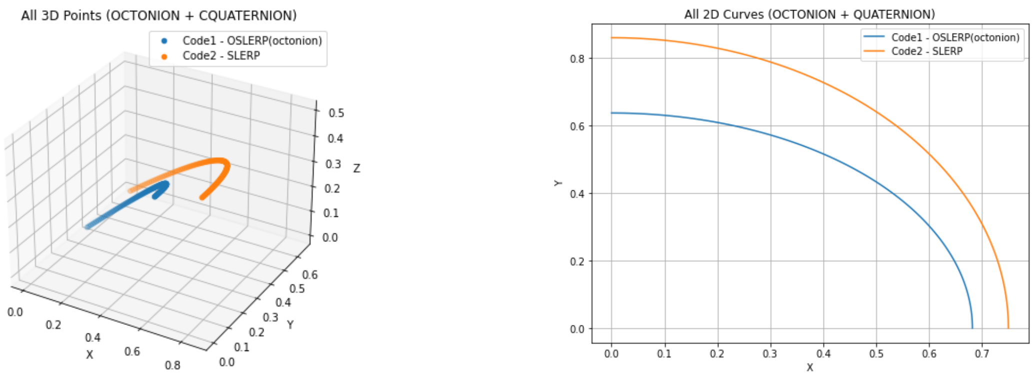

Figure 1. Graphs obtained in the octonion and quaternion spaces in the case where only two control points are available

3.2. THE INTERPOLATION CURVE WITH MORE THAN TWO CONTROL POINTS

Cubic curves are widely used when interpolating through a series of points. Two principal categories of such curves are splines (derived from numerical techniques) and Bézier curves (derived from geometric ideas). A cubic curve can be constructed either using the de Casteljau procedure or transforming a unit square, together with a quadratic curve, onto a warped quadrilateral. In this type of interpolation, two parameters are utilized for parameterization, where the second parameter serves as a blending function $\left(v=2t\left(1-t\right)\right)$ [13].

In the octonion-based spherical framework, the formulation of the spherical cubic interpolation curve has the form:

$$Osb\left(x_i,x_{i+1},y_i,y_{i+1},t\right)=Oslerp\left(Oslerp\left(x_i,x_{i+1},t\right),Oslerp\left(y_i,y_{i+1},t\right),v\right).$$

Theorem 2. The curve created by the Osb must be continuous and differentiable.

Proof: It is enough to check the continuity at the control points x_i (or $Osb\left(x_{i-1},x_i,y_{i-1},y_i,1\right)=Osb\left(x_{i+1},x_i,y_{i+1},y_i,0\right)$).

$$Osb\left(x_{i-1},x_i,y_{i-1},y_i,1\right)=Oslerp\left(Oslerp\left(x_i,x_{i-1},1\right),Oslerp\left(y_i,y_{i-1},1\right),0\right)$$
$$Oslerp\left(x_i,y_i,0\right)=x_i$$

$$Osb\left(x_i,x_{i+1},y_i,y_{i+1},0\right)=Oslerp\left(Oslerp\left(x_{i+1},x_i,0\right),Oslerp\left(y_{i+1},y_i,0\right),0\right)$$
$$Oslerp\left(x_i,y_i,0\right)=x_i$$

in the second step, it suffices to prove that it is continuously differentiable at the control point x_i . For convenience, we put

$$U\left(t\right)=Oslerp\left(x_i,x_{i+1},t\right),\ V\left(t\right)=Oslerp\left(y_i,y_{i+1},1,t\right),\ W\left(t\right)=U\left(t\right)^{-1}V\left(t\right).$$

So,

$$Osb\left(x_i,x_{i+1},y_i,y_{i+1},t\right)=U\left(t\right)W\left(t\right)^{2t(1-t)},$$

$$\frac{d}{dt}Osb\left(x_i,x_{i+1},y_i,y_{i+1},t\right)=U\left(t\right)\left(\frac{d}{dt}W\left(t\right)^{2t(1-t)}\right)+\left(\frac{d}{dt}U\left(t\right)\right)W\left(t\right)^{2t(1-t)}.$$

From proposition 6

$$\frac{d}{dt}W_i(t)^{2t(1-t)} = \Big[-\sin\big((2t(1-t)\theta_{W_i(t)}\big)\big((2-4t)\theta_{W_i(t)} + 2t(1-t)\theta'_{W_i(t)}\big),$$
$$\cos\big((2t(1-t)\theta_{W_i(t)}\big)\big((2-4t)\theta_{W_i(t)} + 2t(1-t)\theta'_{W_i(t)}\big)b_{W_i(t)}$$
$$+\sin\big(2t(1-t)\theta_{W_i(t)}\big)b'_{W_i(t)}\Big].$$

Now, for $x_i \in O_1$, y_i is as follows so that, the condition of differentiability is established

$$\frac{d}{dt}Osb(x_{i-1}, x_i, y_{i-1}, y_i, 1) = \frac{d}{dt}Osb(x_i, x_{i+1}, y_i, y_{i+1}, 0).$$

Then

$$x_i\left(\log\left(x_i^{-1}x_{i+1}\right) + 2\log\left(x_i^{-1}y_i\right)\right) = x_i\left(\log\left(x_i^{-1}x_{i+1}\right) - 2\log\left(x_i^{-1}y_i\right)\right)$$
$$4\log\left(x_i^{-1}y_i\right) = \log\left(x_{i-1}^{-1}x_i\right) - \log\left(x_i^{-1}x_{i+1}\right)$$
$$x_i^{-1}y_i = \exp\left(\frac{\log\left(x_{i-1}^{-1}x_i\right) - \log\left(x_i^{-1}x_{i+1}\right)}{4}\right)$$
$$y_i = x_i\exp\left(\frac{\log\left(x_{i-1}^{-1}x_i\right) - \log\left(x_i^{-1}x_{i+1}\right)}{4}\right).$$

For this, it is necessary that y_i be determined as

$$x_i\exp\left(-\frac{\log\left(x_i^{-1}x_{i+1}\right) + \log\left(x_i^{-1}x_{i-1}\right)}{4}\right).$$

In the second step, it is enough to prove that x is continuously differentiable at the control points. □

The difference between the resulting graphs in octonion and quaternion spaces is noticeable in Fig. 2.

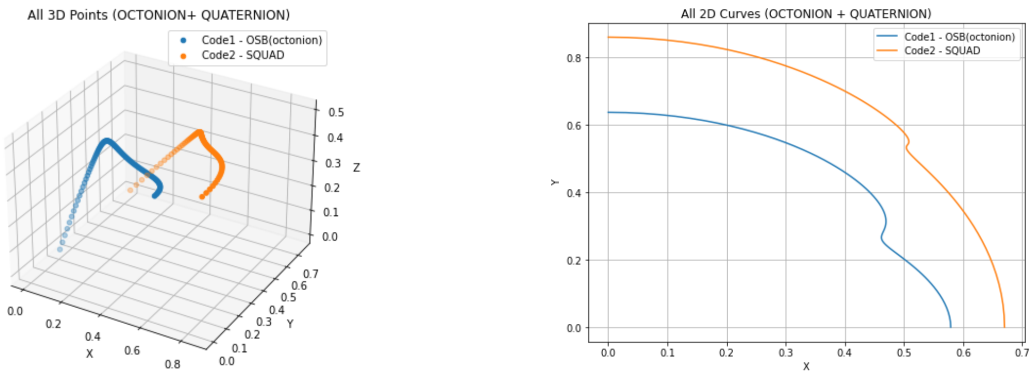

Figure 2. Graphs obtained in octonion and quaternion spaces in the case where more than two control points are available.

3.3. INTEGRATION WITH NUMERICAL APPROXIMATION METHODS

To enhance interpolation techniques in the octonionic context, we extend the approach by combining numerical methods and employing the gradient descent algorithm. Based on differential-geometry principles, the curvature of a curve can be evaluated by analyzing the changes in the vector of velocities.

Definition 3. Suppose that $\beta(s)$ is a parametrized curve with the arc length parameter on the octonion unit sphere. Then the local curvature of the curve $\beta(s)$ is as follows:

$$K(\beta,s)=\left\|\beta''(s)-\left(\beta''(s)\cdot\beta(s)\right)\beta(s)\right\|. \tag{21}$$

As for control points $X_1,...,X_n \in O_1$, we are looking for $x_1,...,x_n \in O_1$ such that $x_{is}=X_s$, (for $s=1,\dots,n$) and the following expression is minimized:

$$E=\sum_{i=1}^{n}A(x_i)\left\|\tilde{K}(x_i)\right\|^2 \tag{22}$$

where $A(x_i)=\dfrac{\|x_i-x_{i-1}\|+\|x_i-x_{i+1}\|}{2}$ and $\tilde{K}(x_i)=x_i''-\dfrac{x_i''\cdot x_i}{x_i\cdot x_i}x_i$ the discrete version of $K(\beta,s)$.

Gradient Descent is an optimization algorithm for finding the local minimum of function E. It is essential to ensure that this version of the interpolation curve is located in O_1. By defining the function $f(x)=x\cdot x-1$, the discrete version of the set of unit octonions is rewritten

$$\begin{aligned} &O_1=\{x\in O \mid f(x)=0\},\\ &G=\sum_{i=1}^{n}A(x_i)\left\|\tilde{K}(x_i)\right\|^2+\delta f(x_i)^2,\ \delta\in\mathbb{R}. \end{aligned} \tag{23}$$

δ should be considered so large that the function G covers at least the function E normally and relatively, and also includes only the unit octonions. Our goal is to calculate $\dfrac{\partial G}{\partial x_{ij}}$ where x_{ij} represents the j-th component of the i-th octonion.

$$\begin{aligned}\frac{\partial G}{\partial x_{ij}} &= \frac{\partial}{\partial x_{ij}}\left(\sum_{i=1}^{n} A(x_i)\left\|\tilde{K}(x_i)\right\|^2 + \delta f(x_i)^2\right)\\ &= \frac{\partial}{\partial x_{ij}}\Big(A(x_{i-1})\tilde{K}(x_{i-1})\cdot\tilde{K}(x_{i-1})\\ &\quad + A(x_i)\tilde{K}(x_i)\cdot\tilde{K}(x_i)\\ &\quad + A(x_{i+1})\tilde{K}(x_{i+1})\cdot\tilde{K}(x_{i+1}) + \delta f^2(x_i)\Big)\end{aligned}$$

The terms of the above equation are calculated as follows:

$$\frac{\partial f}{\partial x_{ij}} = (x_i\cdot x_i) = 2x_i\cdot\frac{\partial x_i}{\partial x_{ij}} = 2x_{ij}$$

$$\begin{aligned}\frac{\partial A(x_i)}{\partial x_{ij}} &= \frac{\partial}{\partial x_{ij}}\left(\frac{\|x_i - x_{i-1}\| + \|x_i - x_{i+1}\|}{2}\right)\\ &= \frac{1}{2}\frac{\partial}{\partial x_{ij}}\sqrt{(x_i - x_{i-1})\cdot(x_i - x_{i-1})} + \sqrt{(x_i - x_{i+1})\cdot(x_i - x_{i+1})}\\ &= \frac{1_j\cdot(x_i - x_{i-1})}{2\|x_i - x_{i-1}\|} + \frac{1_j\cdot(x_i - x_{i+1})}{2\|x_{i+1} - x_i\|}\end{aligned}$$

$$\begin{aligned}\frac{\partial x_i''}{\partial x_{ij}} &= \frac{\partial}{\partial x_{ij}}\frac{x_{i-1} - 2x_i + x_{i+1}}{l(x_i)^2}\\ &= \frac{A(x_i)^2\frac{\partial}{\partial x_{ij}}(x_{i-1} - 2x_i + x_{i+1}) - (x_{i-1} - 2x_i + x_{i+1})\frac{\partial}{\partial x_{ij}}A(x_i)^2}{A(x_i)^4}\\ &= \frac{2A(x_i)^2 1_j + 2(x_{i-1} - 2x_i + x_{i+1})A(x_i)\frac{\partial}{\partial x_{ij}}A(x_i)}{A(x_i)^4}\end{aligned}$$

$$\begin{aligned}\frac{\partial\tilde{K}(x_i)}{\partial x_{ij}} &= \frac{\partial}{\partial x_{ij}}\left(x_i'' - \frac{x_i''\cdot x_i}{x_i\cdot x_i}x_i\right)\\ &= \frac{\partial x_i''}{\partial x_{ij}} - \frac{\partial}{\partial x_{ij}}\left(\frac{x_i''\cdot x_i}{x_i\cdot x_i}\right)x_i - \frac{x_i''\cdot x_i}{x_i\cdot x_i}1_j\\ &= -\left(\frac{\left(\frac{\partial x_i''}{\partial x_{ij}}\cdot x_i + x_i''\cdot\frac{\partial x_i}{\partial x_{ij}}\right)x_i\cdot x_i - 2\left(x_i\cdot\frac{\partial x_i}{\partial x_{ij}}\right)x_i''\cdot x_i}{(x_i\cdot x_i)^2}\right)x_i - \frac{x_i''\cdot x_i}{x_i\cdot x_i}1_j + \frac{\partial x_i''}{\partial x_{ij}}\end{aligned}$$

$$\frac{\partial\tilde{K}(x_i)}{\partial x_{ij}} = -\left(\frac{\left(\frac{\partial x_i''}{\partial x_{ij}}\cdot x_i + x_i''\cdot 1_j\right)x_i\cdot x_i - 2(x_i\cdot 1_j)x_i''\cdot x_i}{(x_i\cdot x_i)^2}\right)x_i - \frac{x_i''\cdot x_i}{x_i\cdot x_i}1_j + \frac{\partial x_i''}{\partial x_{ij}}.$$

The Gradient Descent algorithm is as follows:

1. The first step: Initial guess for point $x_0 = (x_1, ..., x_n)$. In this step, Osb performs better than Oslerp because it converges faster.

2. The second step: We repeat on the initial guess $x_{i+1} = x_i - \mu \nabla G$ where

$$\nabla G = \left(\left(\frac{\partial G}{\partial x_{11}}, \frac{\partial G}{\partial x_{12}}, \frac{\partial G}{\partial x_{13}}, \frac{\partial G}{\partial x_{14}}, \frac{\partial G}{\partial x_{15}}, \frac{\partial G}{\partial x_{16}}, \frac{\partial G}{\partial x_{17}}, \frac{\partial G}{\partial x_{18}} \right), ..., \left(\frac{\partial G}{\partial x_{n1}}, \frac{\partial G}{\partial x_{n2}}, \frac{\partial G}{\partial x_{n3}}, \frac{\partial G}{\partial x_{n4}}, \frac{\partial G}{\partial x_{n5}}, \frac{\partial G}{\partial x_{n6}}, \frac{\partial G}{\partial x_{n7}}, \frac{\partial G}{\partial x_{n8}} \right) \right).$$

3. The last step: Continue the second step until you reach the termination condition. The difference between graphs in octonion and quaternion spaces can be seen in Fig. 3.

The graphics obtained from this method are generated using Python software.

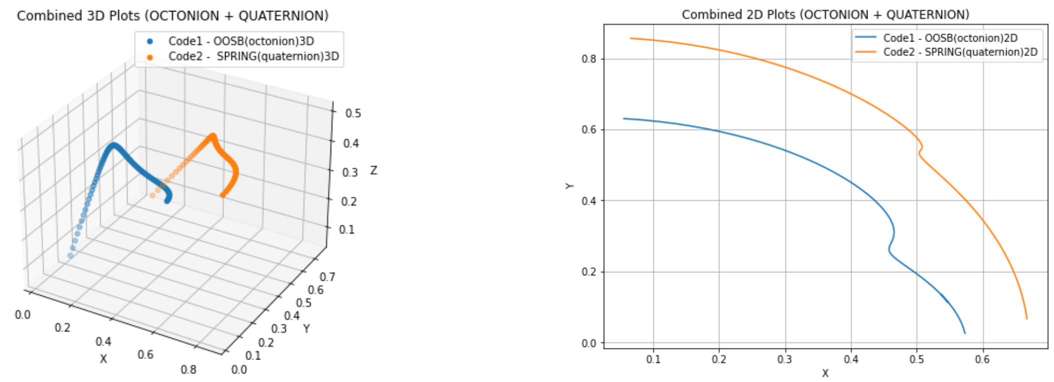

Figure 3. Graphs obtained in octonion and quaternion spaces with the numerical approximation method.

4. CONCLUSIONS

Hamilton's formulation of the quaternionic system in 1843 fundamentally transformed the calculation of three-dimensional rotations. Such rotations are fundamental to describing rigid-body motion and have become core techniques in computational sciences, including computer graphics, robotics, and related areas.

By ensuring differentiability, the interpolation curves obtained in the octonion space show reduced curvature compared to their quaternionic counterparts. Furthermore, when numerical techniques are incorporated into the interpolation framework, convergence accelerates in the octonionic setting, enabling the method to reach a result in fewer computational cycles.

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